Network model for feedback circuits

Douglas R. White\textsuperscript{1,2} Constantino Tsallis\textsuperscript{1,2}
Doyne Farmer\textsuperscript{2} Scott D. White\textsuperscript{4} Natasa Kejzar\textsuperscript{2,5}

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\textsuperscript{1}Institute of Mathematical Behavioral Sciences, UC Irvine, \textsuperscript{2}Santa Fe Institute, \textsuperscript{3}Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, \textsuperscript{4}Information and Computer Science, UC Irvine, \textsuperscript{5}Social Sciences, University of Ljubljana

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Abstract

We investigate a simple agent-based model of information networks whose nodes can emit tokens and route them along edges. Tokens carry information on the identity of their originator and maintain a time-to-live parameter indicating the maximum number of hops the token can still traverse before being dropped from the network. In our model, feedback occurs when a token reaches a responding node and a new edge is created between the token originator and the responding node. By creating an edge in this way, an elementary feedback circuit (ring) in the network is formed. With this model, attachment, distance and traversal parameters are capable of generating many observed network topologies, with variations in clustering, small-world, searchability, and structural cohesion. We explore the extent to which broad classes of network growth governed by these three parameters show several universality classes with q-exponential properties of the degree distribution of nodes. This provides useful q-entropic indices for the study of dynamical behavior and path dependency in network evolution. This simulation belongs to a class of phenomena that we are investigating under the rubric of ring cohesion, which we conceptualize as an important source of self-organization in networks.
We develop what might be considered an abstract agent-based model (ABM) with agents that have a preference to traverse their network in trying to satisfy the goal of locating another agent with whom to form a new tie by sending recruitment tokens through the network until a partner is found or the search fails. Failing the network search they recruit and attach to a new partner not previously connected to the network. Single agents are selected iteratively and each in turn has the opportunity to find either a partner within the network or a new partner from outside, thereby growing the size of the network. The model has three parameters: agent activity, which governs the chances the agent has to find a partner; a distance needed – by the agent’s token which must traverse within the network – to find a partner with whom a new edge can be formed; and a regulator of how the token traverses the search through successive nodes in the network until it expires at the selected distance required to search for a network partner.

Financial markets, intercorporate networks, and marriage networks are but a few of many examples (e.g., metabolic processes) in which the kind of feedback that we model plays a crucial role in the higher-order functioning of the system, including self-organization [1]. Surprisingly, there are relatively few generative mathematical models of feedback networks. A notable exception are network models of catalysts [2], in which self-organization takes off when a series of processes creates autocatalytic cycles that can crash – when one or more keystone nodes are removed – or can lose the capability of dynamic interaction when the network becomes too connected. In these models, nodes come in different catalytic types and quantities, differentiated by patterns of incoming and outgoing arcs, and they multiply in quantity according to the number of incoming arcs that catalyze them.

Feedback networks – information networks that contain feedback circuits – are ubiquitous in nature and society. At a more abstract level, ours is a general model of feedback networks\(^1\) designed to help understand how the phenomenon of feedback creation at the micro-level leads to changes in network topology at the macro level, and how that topology can alter micro-level behavior and feedback. In our model, we represent the flow of information

\(^1\)More general in that there is no traversal in the catalysis model, as with tokens ported from node to node along existing edges. Catalysis is modeled by outwardly directed arcs having catalyzing effects that affect the population indices associated with the node-types they connect to. Catalysis is an important kind of process that can generate feedback once cycles are formed, but token transmission in our model can create feedback cycles independently of direct catalytic effects.
(e.g., written message, currency, trip-taking, etc.) via the abstraction of a token that travels through a network. A token is emitted from a given node with some pre-defined probability distribution and is routed along successive edges according to another random process until a receiver node responds by forming a new edge directly with the original sender, or until a new node, not connected in the existing network, is recruited with which to form an edge. In either case a response is sent back along the new edge. A circuitous response is possible, for example, if the token carries information on the identity of the originator enabling the receiver to contact the original sender [3].

By modeling the evolution of feedback networks in this way, we develop a simple, yet powerful framework in which common characteristics of many different types of evolving feedback networks can be examined. It is by including the effects of network growth processes based on information transmission and cycle formation (which allows the feedback) that we can model the co-evolution of a network’s topology and the micro processes operating within it.

The occurrence of response cycles may be important even if they operate only once, or, once established, depending on the phenomenon, the circuits created by such responses may be used over and again until one or more of their edges dissolves or one of the nodes is removed. Our model enables us to distinguish which edges were initially generated in the process of cycle formation and which were not, at successive time periods in network evolution.

Substantively, we classify these two types of nodes as consolidating ties – those generated by the feedback process – and novelty ties, those formed by recruitment of a new node and edge. Simulations [24] and empirical network studies [11], [25] argue for the importance and complementary of consolidation and innovation in social networks and organizations.

Simple as it is, this model is sufficiently general as to capture many of the important characteristics of feedback networks that exist over a wide range of complex systems. Even an abstract market, for example, will operate to create feedback cycles that consummate transactions (seller – ask – buyer – bid – seller, for example). Empirical market processes typically operate on longer chains that transmit tokens, whether information, goods, currency, or credit, whose circulation can be transacted. Additional features and con-

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In a variant of this model the token might also carry an address or description of an intended receiver, and is successful when an intended receiver responds to the originator. Such variants may help to explore the searchability of the network topologies that we generate with the more generic cycle-generating model.
straints can be added within the framework of this general model that allow additional testable hypotheses with more specialized models.

This feedback-driven model of network processes and parameters is motivated by empirical research on social networks of several varieties. One is the study of networks of trade relations among merchants in different cities [4], where the tokens might represent information (e.g. written messages) sent by merchants that circulate to near or distant places or cities where transactions may be consummated. They might also represent the travel of agents sent to find partners with whom to transact business on a merchant’s behalf. The counter flow of goods, credit, or payment for transactions completed might constitute the cycles constructed in this model, as would establishment of continuing relations to new partners located in the network. A second type of study [5] where the importance of cycles is demonstrated is the domain of kinship and marriage networks. These studies establish that it is structurally endogamous cycle formation in marriage networks, where people marry someone already connected through a chain of blood or marriage ties, that predicts the boundaries of class formations, ethnicities, religious groups, groups that differentially transmit wealth and political influence, and thus the social boundaries of elites. Here, information and agents circulate in micro time to contract marriages. In some cases there is characteristic evidence that the self-organization of marriage linkages follows power-law frequency distributions. This occurs when the orderings of the frequency distributions follow the diameter and linkage bias of the cycles.

A third organizational domain of study where the importance of social cohesion created by linkages based on multiple connectedness (implying cycles) has been demonstrated is biotechnology [7], [25].

Our model endogenizes the balancing of saturation/innovation (consolidation with feedback cycles; novelty with new tie formation) [24] in ways that may match not only the microdynamics of tie formation with networks but also the resultant variation in macro level network typology. In all three types of networks studied – biotech, marriage, and merchant trading – there is reaching out to form ties with new partners outside the network. In marriage networks the choice is between a structurally endogamous marriage [5] and marriage with an outsider. In biotech [11] the choice is between high redundancy choices between firms in the cohesive core versus reaching out to form a high-innovation tie with new entrants to the industry. For merchant traders [4] there is high reliability and volume for trades with other merchants already cohesively linked but perhaps lower profit in wholesale trade,
but a concomitant necessity to bring in new higher risk innovative ties with outside or new retail clients.

The study of cycle formation is analytically difficult, and thus lending itself to simulation, because the formation of an edge that completes a cycle is not independent of the other edges. Our objective, however, is not simply to use Monte Carlo simulation as a means of gaining insights about network evolution. Rather we want also to capitalize on the very nonindependence of network phenomena that makes the study of cycle formation so difficult. We explore the applicability of a generalized statistical mechanics that applies to large ensembles of nonindependent as well as independent events, provide analytic foundations for our study. To do so we use mathematical techniques connected with q-entropy, in which interactions may be independent or non-independent [9], [10], [11].

More generally, we are interested in the statistical mechanics by which micro behavior in the network is aggregated into global properties of the network, and how such global properties implicate effects on the structure of local contexts and hence on local behavior [11]. If all interactions were simple equiprobable and independent processes, for example, we would expect standard entropy equations to apply. Because network processes may be contingent on growing inequalities in local network structure, we are especially interested in how processes reverberate through a network to produce steady-state outcomes that belong to universality classes of nonextensive statistical mechanics which are being newly discovered for nonindependent interactions [9], [12].

The General Model. We model feedback network by an evolving graph having nodes that can emit tokens that can traverse the edges of the graph in either direction, thus making the underlying graph undirected. In this representation, tokens carry information on the identity of their source and maintain a time-to-live parameter (TTL) indicating the number of hops (edges) the token must traverse before being dropped. If the token manages to reach a node at a TTL distance appropriate to qualify it as a responding node, a response is initiated. In our model, rather than route the response back along the original path, as in computer network routing, a response is made by creating a new edge between the source and the TTL-qualified target.  

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3In our model this occurs at TTL expiration, but in some applications the responder node may be targeted, in others self-selected. In still others information carried by the token may be intended to evoke response from an appropriately matching node.

4Note here the difference between our model and the constraints of routing theory as
This consolidating edge thus creates a feedback circuit that we designate as an \textit{elementary cycle} in the evolving network.\footnote{In marriage networks, the consolidating edge corresponds to a relinking marriage.} In contrast, a \textit{novelty edge} is created between the source node and a new target node previously unconnected to the network only when a token cannot find a suitable target within the network that is not already connected to the source node in exact TTL distance. We assume an unlimited pool of outside nodes available for novelty edges.

We simulate network growth and cycle formation by a random iterative process that involves time steps with three elements, each representing a stage in network transmission, and with the third culminating in modification of the network by addition of a new edge, possibly associated with a new node. Each process element – the generation and assignment of a token to a source node (representing an agent) in the network, the transmission of the token, and the routing of the token through the network – is characterized by a structural characteristic of the network and an associated parameter (exponent) that combine to affect the probability of this type of event.

Each structural characteristic in the model – the number of edges of a node (degree), the number of edges that the token must traverse (TTL distance) to find a target node within the network, and the number of nodes still untraversed by the token from its current location if its TTL is not reached – has an associated parameter exponent. There are two reasons for using exponential parameters. One is that exponents of zero for a potentially biasing variable in a generating probability function (or approaching 1 in the case of inverse power-law distance decay) correspond to equiprobability. This allows us to contrast the effects of equiprobability \cite{16} with biased probability in our generating functions for network growth.

Another reason for using exponents as parameters in our model is that power laws are the necessary and sufficient condition for scale-free behavior \cite{13}, in which changing the scale of an independent variable $x$ does not change
the functional form of a dependent variable \( y = f(x) \), i.e., the shape of the distribution of variation in \( y \) relative to \( x \).

The scale-free property has been used to great advantage by Barabási and colleagues [14] in the study of networks. They define scale-free [SF] networks as those in which attachment of a new node to an existing one is proportional to the latter’s degree. In Barabási’s SF networks the probability of attachment to a node \( i \) is directly proportional to its degree, \( P(i) \propto \text{deg}(i)^c \), where \( c = 1 \) is the exponent. The scale-free network resulting from this preferential attachment will have a degree distribution with power-law coefficient \( \lambda \), however, in which \( \lambda \to 3 \), from below, as \( n \to \infty \) [15]. The SF models lack any sense of agent-based behavior for the simple reason that there is no concept of a local neighborhood. The attractive probability is global, as if every new node attached with full cognizance of the current degree of every node of the network. There are no local gradients that govern behavior. The same is true of the preferential attachment growth model [9] of Soares et al. of which SF-networks are a special case. Neither of these models generates networks with the properties of coherent fields [25].

The three probability functions for generating of token traversal and network growth in our model are these:

- **Preferential attachment of a token to a node.** For each time step a single token originates by attachment to a single source node \( i \) sampled with probability \( P_\alpha \), proportional of its degree \( \text{deg}(i) \) raised to power \( \alpha \), the attachment parameter:

  \[
  P_\alpha(i) = \frac{\text{deg}(i)^\alpha}{\sum_{j=1}^{N-1} \text{deg}(j)^\alpha}
  \]

  Because the denominator is a constant normalizing term, attachment is directly proportional to degree if \( \alpha = 1 \) and equiprobable if \( \alpha = 0 \):

  \[
  P_\alpha(i) \propto \text{deg}(i)^\alpha
  \]

- **Token is assigned a maximum distance-to-traverse (TTL) with a probability that decays with the distance.** Each token,
at the time of its origination, is assigned a maximum distance \( d \) to travel, from node \( i \) to successive nodes within the network, without backtracking or traversing any node twice. A distance decay TTL is an integer distance \( d \) sampled with probability \( P_\beta \) where \( \beta > 1 \) is the distance decay parameter: \(^7\)

\[
P_\beta(d) = \frac{d^{-\beta}}{\sum_{k=1}^{\infty} k^\beta} = \frac{d^{-\beta}}{\text{zeta}(\beta)}
\]  

(3)

Because \( \text{zeta}(\beta) \) for \( \beta > 1 \) is a constant normalizing term, the probability of a TTL \( d \) decays proportionally with distance \( d \):

\[
P_\beta(d) \propto d^{-\beta}
\]  

(4)

Thus, a token that originates from \( i \) is transmitted in successive hops until the time step terminates when either the TTL reaches 0 or there is no adjacent node that has not already received the token. In the first case, a response is sent from the recipient back to the source node and a corresponding edge is created to complete a source-recipient cycle. Where, in the second case, for a token whose TTL is not expired, there remains no unvisited recipient from its current node, the token is dropped. A new node is recruited from outside the network to receive the dropped token and an edge is created between the source node and the newly recruited node. We assume an unlimited number of recruitable nodes lying outside the network.

- **Look-ahead successive node routing for tokens.** A node \( j \) that initiates or receives a token with an unexpired TTL samples one of its neighbors \( l \) to receive it with probability \( P_\gamma \), proportional to one plus unused degree \( u(l) = 1 + \text{number of } l\text{’s neighbors that would still be unvisited from } i, j, l \text{ is taken from } i \text{ through } j \text{ to } l \). \(^8\) Here, \( m = \{1 \ldots M\} \) designates the set of neighbors of

\(^7\)The denominator of equation (3) is the sum of a generalized harmonic series that is convergent to a finite limit only for \( \beta > 1 \), in which case the sum corresponds to the zeta function.

\(^8\)For the starting node, this corresponds to the degree of adjacent nodes, but can only diminish as paths are traversed.
l, with \( P_{\gamma}(l) = 0 \) if \( _{i,j}u(l) = 0 \), i.e., for a successor \( l \) previously visited, and \( \gamma \) is the routing parameter:

\[
P_{\gamma}(j, l) = \frac{(1 + _{i,j}u(l)\gamma)}{\sum_{m=1}^{M} (1 + _{i,j}u(m)\gamma)}
\]

(5)

Because the denominator is a normalizing constant, routing is equiprobable among unvisited neighbors of \( j \) if \( \gamma = 0 \). If \( \gamma > 0 \) then node \( j \) samples a routing probability based on a look ahead to which neighbors have higher unused degree and act as local hubs. Selection probability of neighbors is proportional to \( _{i,j}u(l) \), the unused degree of \( l \):

\[
P_{\gamma}(j, k) \propto _{i,j}u(l)^\gamma
\]

(6)

Traversal bias does not affect the degree distribution of our evolving graphs, but like the effects of major highways, it produces a greater abundance of 'side roads' that are offshoots of the most traversed edges. We should therefore investigate its effects on network topology (outcome graphs) in terms of edge centrality and of cycles with high edge centrality.

**Outcome Graphs.** Typical feedback networks with \( N = 250 \) for \((\alpha, \beta, \gamma)\) of \{\((0, 1.3, 0), (0, 1.3, 1), (1, 1.3, 0), (1, 1.3, 1)\)\} are shown in Figures 1 and 2. We refer to our outcome graphs as t-graphs, for token-graphs.

The two figures display same four graphs. In Figure 1 the sizes of the nodes represent their degrees and in Figure 2 the thickness of the edges is proportional to the number of successfully created feedback cycles in which the edge participated. We can see the distribution of edge weights (calculated on 100 realizations of networks growing to \( N = 500 \)) for parameter \( \beta = 1.3 \). We can see that parameter \( \gamma \) has the largest effect on edge weights.

The preferential attachment \( \alpha \) bias generates a range of hub topologies, some of which lower average distances and some which, when hubs tend to be connected, allow easier searchability [20]. Larger parameter \( \beta \) results in denser networks (especially when coupled with large parameter \( \gamma \)) since due to small possibility to generate large TTLs mostly short feedback cycles can be formed. For larger parameter \( \gamma \) due to greater clustering of the nodes the structure of the networks diverge from tree-like structure.

Because cycle formation occurs between pairs of nodes already connected it represents nonindependent phenomena. We study low-level topological
Figure 1: Representations of typical network models with 250 nodes for $\beta = 1.3$. Models: (a) $\alpha = 0, \gamma = 0$, (b) $\alpha = 0, \gamma = 1$, (c) $\alpha = 1, \gamma = 0$ and (d) $\alpha = 1, \gamma = 1$. Sizes of nodes are proportional to their degrees. In bottom graphs hubs emerge spontaneously due to preferential attachment ($\alpha = 1$) while more clustering occurs because of larger routing parameter in cycle formation ($\gamma = 1$).
Figure 2: Representations of typical network models with 250 nodes for $\beta = 1.3$. Models: (a) $\alpha = 0, \gamma = 0$, (b) $\alpha = 0, \gamma = 1$, (c) $\alpha = 1, \gamma = 0$ and (d) $\alpha = 1, \gamma = 1$. Thicknesses of edges are proportional to their weights (the number of successfully created feedback cycles (minus one) in which the edge participated).
Figure 3: Distribution of edge weights. Edge weights represent the number of successfully created feedback cycles (minus one) in which an edge participated.
property – node degree distribution of simulated networks. We simulated 10 realizations of networks with $N = 5000$ nodes for different $\alpha, \beta$ and $\gamma$ model parameters. Examples of our numerical results for degree ($k$) distribution $p(k)$ for some of the models are indicated in Figure 4. The values of the fitting parameters can be seen in Table fitted values. We illustrate the fact the degree distribution is changing according to all three parameters. The solid curves on the graphs in Figure 4 represent q-exponential fitting with the form:

$$p(k) = p(0) x^\delta e_q^{\frac{k}{\kappa}}$$

(7)

where the q-exponential function is defined as follows

$$e_q^x = [1 + (1 - q) x]^{1/(1-q)} \quad (e_1^x = e^x)$$

(8)

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<th>Fitted parameters</th>
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Table 1: Parameters of the fitting q-exponential curves for the network models specified. The dots represent the actual fitted values.

Why have we fitted the model degree distributions with q-exponentials? (Queiro)

**Fittings of parameters.** We estimated q-exponential parameters for various different model parameters $\alpha = \{0, 0.25, 0.5, 0.75, 1\}, \beta = \{1.1, 1.2, 1.3, 1.4, 1.5\}$
Figure 4: Degree distributions and q-exponential fitting for simulations (5000 nodes, 10 realizations). (a) models with $\beta = 1.2, \gamma = 0$ and varying $\alpha$ ($\{0, 0.5, 1\}$ colors black, blue and red respectively), (b) models with $\beta = 1.2, \gamma = 1$, (c) $\beta = 1.4, \gamma = 0$ and (d) $\beta = 1.4, \gamma = 1$ (with same varying $\alpha$). Consider x-axes are in different scale. The parameters of the fitting q-exponential curves can be seen in Table.
and $\gamma = \{0, 0.5, 1\}$. Figures 5 and 6 show the dependences of parameters of best fits to model parameters. We can see that parameter $\delta$ depends solely to model parameter $\alpha$. The other two q-exponential fitting parameters ($q$ and $\kappa$) on the other hand depend on all three model parameters. We can see the divergence for parameter $\kappa$ when $\beta$ and $\gamma$ grow and $\alpha = 0$. We can also notice the steep growth of $q$ parameter with growth of all three model parameters.

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Figure 6: Dependencies of q-exponential parameters $q$ and $\kappa$ that were fitted to network models.


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