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shown in Figure 6.[4] It has at most two node-independent *s-t* geodesics, one of which may traverse one of the thicker *s-t* paths along the left and upper sides of the graph, while another geodesic may traverse the one of the thicker paths along the lower and right sides. No more than two such geodesics will have no nodes in common other than s and t. With the exception of one edge, circled in Figure 6, this graph has moiety properties A, B and C, which are rare in the family of all s-t graphs, but are properties that dispose to the collective discovery of geodesics.

Conclusion

Combining Theorems A, B and C, we link moiety properties of the global structure of graphs to the g- and g+ inequality as local traversal frequency properties on edges. This gives upper and lower bounds for the g- and g+ inequalities. Moiety property A-B-C implies the g+ inequality, which implies g-inequality, which in turn implies moiety property A-B.

The definitions and theorems presented here give some preliminary indications about how collective solutions to maze-traversal by the marking of paths traversed can co-evolve with the structural properties of environments and strategies of maze traversal. In the sequel we plan to extend these results to probabilistic models of maze traversal, provide further applications and examples, some from business organization, and show how simulation results support further propositions as to the structural properties of graphs that make collective discovery of geodesics possible.

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[4] The graph in Figure 8 is a single block, so is not decomposable by cutnodes. Another way to analyze the structural properties that contribute to or detract from collective discovery of geodesics in parts of an *s-t* graph is to use the *segmentation* of *G* by removal of a separated subgraph *S'* defined by an *s'-t' segment* of *G* that is a subgraph of G containing a maximal g-path *P* of *G* and a subgraph *S'* separated from *G* by removal of the *s'-t'* end-nodes of *P*. By successive segmentation and removal of separated subgraphs of *G* we may analyze successive segments in terms of their structural properties. The properties of the segments will contribute to those of the *s-t* graph. This decomposition will be used in our sequel to analyze Johnson's graph, segment by segment, with regard to probabilistic and simulated maze traversals and collective discovery of geodesics.

Theorem C. Every *s-t* graph *G* that satisfies g+ inequality has the moiety A-B-C-property.

Proof. It is sufficient to prove this for a cohesive *s-t* block *B* of *G*. Let an *s-t* block *B* have g+ inequality: given previous theorems, there is some g-edge e_4 such that f(e_4) > minimum f(e_1) = maximum f(e_2) for all g-edges e_1 and ~g edges e_2; and since g+ implies g-inequality, then by Theorems A and B, *B* must have the moiety A- and B- properties. Assume that *B* lacks the moiety C-property so that there is a set *S* of *k* node-independent *s-t* paths in *B* in which one or more paths is not a geodesic, hence containing a ~g edge e_3. By the moiety A- and B- properties: There are only two such paths in *S*, one of which must be a geodesic because in any set of node-independent *s-t* paths at least one is an *s-t* geodesic. Now there is no ~g edge e_2 for which f(e_2) is greater than f(e_1) for a g-edge e_1. By g+ inequality, f(e_4) for some g-edge e_4 is greater than f(e) for any ~g edge $e = e_3$ or e_2. But if a ~g edge *e* is on one of no more than two node-independent *s-t* paths in *B*, then by moiety A-property the f(e) for all such ~g edges *e* must increase as fast as the f(e_1) for e_1 edges in the geodesic, hence a contradiction.

Applications

Two concrete examples will suffice to show that the moiety properties defined here are of special interest as conditions for collective discovery of geodesics to occur in maze or generalized traversal problems. First, in ant behavior: As available food sources become more distant and the density of food searchers thins, forager ants traverse the environment in straighter lines with occasional crossing of paths (Gordon 1999:108), so the graph of the trails produced by this behavior starts to resemble one with moiety property A. Collective marking of paths traversed produces under these conditions a greater likelihood of marking paths that serve as geodesics.

The final example returns to Johnson's (2000, 2001) simulations on the extent of collective discovery of geodesics in simulated mazes. The maze used in his experiments was another node-independent *s-t* path of *B*, as in graph 2a, can the increase of f(e_1) for g-edges e_1 keep pace with that of f(e_2). Any example with this property will generate this contradiction since by the Subgraph Lemma f(e_1) cannot increase faster than f(e) by the addition of other ~g edges.

The moiety A-B-property is again necessary but not sufficient for g-inequality: graphs 3a, 3b, 3d and 3g have g-inequality and the moiety A-B-property, but if we add a node and path between *v* and *t* in 3a to define a new graph 3a', this graph has the moiety A-B-property but not g-inequality.

An *s-t* block *B* in which *k* is the maximum number of node-independent *s-t* paths has *property C* if there is no set *S* of *k* node-independent *s-t* paths in *B* in which one or more of the paths in *S* is not a geodesic. An *s-t* block *B* has *moiety A-B-C-property* if it has the moiety A-B property and property C. Graphs 3a and 3b have this property, which can only be a sufficient condition for g-inequality and a necessary condition for g+ inequality (graphs in which high f(*e*) values identify g-edges). This property helps to identify g as opposed to ~g edges in node-independent *s-t* paths, but not for graphs such as 3g or cohesive block 3d.

Table 1 summarizes the properties of each example in Figure 3 and the discussion.

	TRAVERSAL		PROPERTIES	STRUCTURAL		
Block	g- and g+ inequality	g-inequality: max g min ~g	# of node-independent *s-t* paths	Moiety A-property	Moiety B-property	Moiety C-property
2a	Yes Yes	g5 = ~g5	2	Yes	Yes	Yes
2a'	No No	g5 < ~g6	2	Yes	Yes	No
2b	Yes No	g2 = ~g2	1	Yes	Yes	Yes
2c	No No	g6 < ~g8	2	Yes	No	No
2d	Yes No	g1 = ~g1	1	Yes	Yes	No
2e	No No	g5 < ~g6	3	No	No	Yes
2f	No No	g5 < ~g6	3	No	No	No
2g	Yes No	g2 = ~g2	2	Yes	Yes	No
2h	No No	g5 < ~g6	3	No	No	No
2i	No No	g3 < ~g4	3	No	No	Yes

Table 1. Characteristics of Graphs in Figure 3

property – having only two node-independent geodesics – is involved once again.

The moiety A-property is necessary but not sufficient for g-inequality: In Figure 3, for example, graphs 3a-3d and 3g have g-inequality but 3c lacks it. Another example is Figure 8, which has the moiety-A property but lacks the g-inequality.

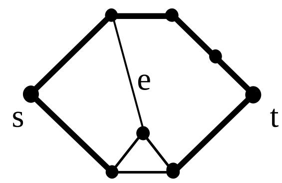


Figure 8: An *s-t* graph that has the moiety-A but lacks the moiety B-property.

A cohesive *s-t* block *G* has *moiety property* B if the interior *C* of *G* contains no cycles with g-edges in a single *s-t* geodesic of *G* and a ~g edge *e* not contained in a cycle of *C*. Moiety property B describes and prevents the situation in graph 3c and Figure 8 in which f(e_2) increases faster than f(e_1) to destroy g-inequality. An *s-t* graph *G* has the *moiety A-B-property* if it has the moiety properties A and B.

Theorem B. Every *s-t* graph *G* that satisfies g-inequality has the moiety B-property, but not conversely.

Proof. We know from Theorem A that an *s-t* graph *G* with g-inequality has the moiety A-property. Assume such a graph does not have the moiety B-property. Then the interior *C* of a cohesive block *B* of *G* contains a cycle with g-edges in a single *s-t* geodesic *D* of *B* and a ~g edge *e* not contained in a cycle of C such that *e* lies on a path *P* connecting a node in *D* to another node-independent *s-t* path. This implies, as in Figure 8 (and graph 3c), that a ~g edge *e* that increases f(*e*) by at least 4 because it connects to the cycle in the interior of *B* that contains a g-edge in *D*, but creates paths that increase f(e_1) for each g-edge e_1 in *D* (or in other node-independent geodesics in which *P* has an endnode) only by 2, thereby violating g-inequality, hence a contradiction for this example. Only if *e* is contained in a cycle with g-edges in

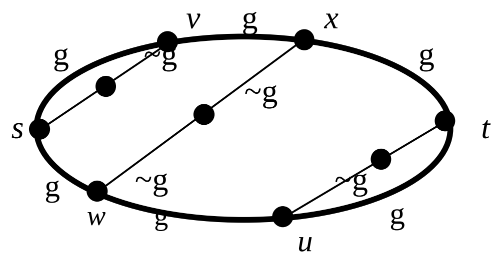


Figure 7: Three types of ~g paths for an *s-t* graph with two node-independent geodesics.

(1). Referring to Figure 7, a path *u*-t increases f(e_2) by 1 for its ~g edges e_2 and f(e_1) by 1 for the g-edges e_1 in the *s-u* part of the geodesic. There is no way that any f(e_1) can grow larger than all f(e_2) by this means.

Similarly for (2) by the *s-t* duality principle.

(3a). Since the geodesics are node-independent, as shown in Figure 7, every such *w-x* path *P* of ~g edges will increase f(e_2) by 2 for each edge e_2 in *P* and increase f(e_1) by 1 for each edge e_1 in the geodesics. If an edge e_2 is in multiple ~g paths that are connected to the same two geodesics, then f(e_2) will increase at least as much as to f(e_1). Otherwise, if multiple ~g paths are connected to different geodesics, then f(e_2) will increase faster than any of the f(e_1). Hence there is no way for f(e_1) > maximum f(e_2) for a g-edge e_1 in *B* by any of the means discussed so far.

(3b). if the geodesics are not node-independent, there are two possibilities:

(a) if the *w-x* path does not connect the geodesics on opposite sites of a node of intersection between them, the increase of f(e_2) relative to f(e_1) is the same as the previous case, or

(b) if not, the *w-x* path connects nodes on opposite sides of an intersecting node between the geodesics, then f(e_2) for the e_2 edges on the *w-x* path will increase more than any f(e_1) for edges in the geodesics. Hence there is no way for f(e_1) > maximum f(e_2) for every g-edge e_1 in *B*. Generalization to an *s-t* graph *G* follows by the g-Inequality Observation.

Similarly for an *s-t* graph which is not a single *s-t* block. The Minmax Theorem tells us the conditions for random path traversal to do as well for G- as for ~g edges, and a moiety-like sics; but this result will hold only for graphs with moiety properties. A large part of the "distributed intelligence" of his result lies in the structure of the maze itself. There is considerable variability to the possible paths that can be taken in his maze, but random paths will traverse the geodesics again and again to, at least as frequently as to the non-geodetic (~g) edges. That is, the "dual organization" of the graph contributes to the collective solution to the problem posed by the maze, which is not solved merely by the laying down of pheromones or trail markers to aggregate collective experience. In fact, if the single edge circled in the lower right is removed from this graph, Theorem A applied to the resultant graph shows it to have one of the necessarily conditions for g-inequality – a potential for collective intelligence – in maze traversal.

We now examine some other properties of the g-inequality and further aspects of "dual organization" in maze (*s*-*t* graph) traversal that are associated with g-inequality.

Minmax Theorem. If a cohesive *s*-*t* block *B* has one or more ~g edges, two node-independent geodesics, ~g edges only on paths between nodes on distinct geodesics, and satisfies g-inequality, then the min {f(e_1) for e_1 in the set E_1 of g-edges} = max {f(e_2) for e_2 in the set E_2 of ~g edges}.

Proof: Each geodesic in an *s*-*t* block *B* increases f(e_1) by one for each of its g-edges e_1. The only other means by which f(e_1) and f(e_2) are increased for g- or ~g edges are by distinct

(1) *u*-*t* paths of ~g edges for *u* in a geodesic in *B*;

(2) *s*-*v* paths of ~g edges for *v* in a geodesic in *B*; or

(3) *w*-*x* paths of ~g edges for *w*,*x* in distinct geodesics in *B*.

These are illustrated in Figure 7.

that the sum of all distinct paths (like pheromones on ant trails) weights heavily on the geodesics, marking out the shortest paths. Before exploring these “moiety” properties more fully, we examine the study and finding that led us to analyze this problem: Johnson’s maze, shown in Figure 6. In this maze, two sets of edges are circled. The large circle touches on edges that run between a geodesic in the upper left, one that shares nodes with many other geodesics in the upper left, and the small circle on the lower right touching on only one edge that connects two adjacent geodesics that are distinct for paths of length 5 out of a total geodesic distance of 9 from start to goal. If we delete that one edge, then this graph would contain no adjacent 20~g paths incident to three distinct *s*-*t* geodesics, which is a precondition of g-inequality.

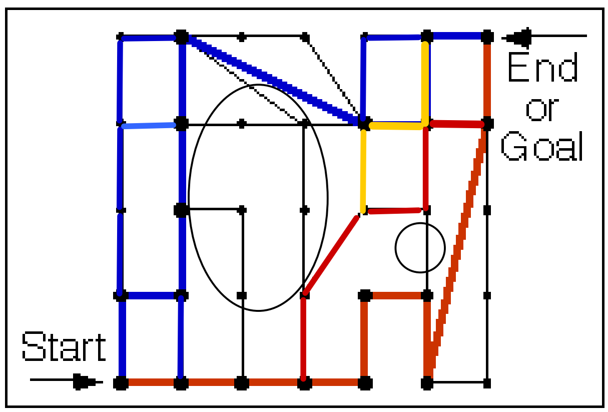


Figure 6: Johnson’s (2000) maze.

Most of the ~g paths in the maze of Figure 6 reinforce path-traversals on the upper left and lower right geodesics, contributing to a “collection solution” to the problem of traversing the maze in the shortest number of steps. Hence we would predict that when Johnson runs his simulation of random traversals on this particular graph (eliminating cycles, so that all traversals are paths), the sum of independent traversals over edges will tend to mark out the geode-

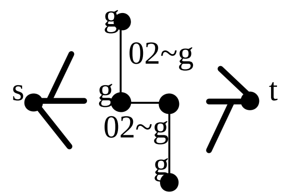


Figure 5: Construction for proof

If an *s*-*t* graph *G* has no adjacent 20~g paths, this implies that all paths or cycles in the interior *C* of *G* have nodes in no more than two *s*-*t* geodesics of *G*. We call this the *moiety A-property*. Graphs 3f and 3i are examples of *s*-*t* graphs that have adjacent 20~g paths and lack the moiety property and Figures 4a and 5 provide graphic examples of having the moiety property.

Discussion. The various versions that we call *moiety properties* have now been seen in Figure 4a and the Switching Theorem, Figures 3a. 3b. 3d and 3g, each of which has two principal *s*-*t* geodesics, and Theorem A, which entails the result that all *s*-*t* graphs with the g-inequality have what we call moiety property A. What is a moiety, and what is the general concept that underlie these various manifestations of a moiety principle, and what are its implications? *Moiety* is derived from *moietie*, a French word meaning “half, ” as in *moietie et moietie* (half and half), and was the label applied to the early French ethnographic descriptions of a common principle of American social organization, the division of a society into two halves for purposes of competitive games, political oppositions, or two exogamous intermarrying groups. It is a form of social organization known as “dual organization.” In the present context it is a property of a graph that simulates a problem-solving maze: it so happens that *s*-*t* mazes having moiety properties – such as graphs in which the non-geodesic paths wander between only two distinct sets of geodesic paths – give rise to the traversal property of the graph but there exist no 11g and no 02g-edges). If the edges of graph 3a are labeled accordingly, they consist entirely of 20g6 and 11~g5 nodes, combining the designation for type of edge (20g and 11~g) with the designation for the frequency f(*e*) of distinct *s-t* paths containing-edge *e*. Hence 20g6 designates a g-edge connecting two g-nodes (20) that are traversed by 6 distinct *s-t* paths. Similarly, we designate ~g paths (those that contain a ~g edge) as 20~g, 11~g, or 02~g. This leaves the g-paths, designated 20g since they terminate in g-nodes. Two paths are *adjacent* if they share a common endnode.

Theorem A: An *s-t* graph *G* with g-inequality has no two adjacent 20~g paths incident to three distinct *s-t* geodesics.

> **Proof**. The proof follows by construction. We prove the logically equivalent contrapositive: an *s*-t graph *G* with adjacent 20~g paths incident to three distinct *s-t* geodesics lacks the g-inequality. Figure 5 shows a general construction for an *s-t* graph with adjacent 20~g paths incident to three distinct geodesics. For each g-edge e_1, f(e_1) is incremented by 1 for each of the three distinct geodesics that contains e_1, and is incremented by 2 for each of the two adjacent 20~g paths, while f(e_2) is incremented by six for each ~g edge e_2 in the two adjacent 20~g paths. If there is one e_1 that lies on a single geodesic, then for that e_1 there exists an e_2 such that $f(e_2) = 6 > 5 = f(e_1)$, and g-inequality is violated. That there is one e_1 that lies on a single geodesic follows from the fact that if there are k g-edges incident to *s* and each such edge lies on two or more distinct *s-t* geodesics, then there must be branches along their geodesics between *s* and *t*, but each branch by the same criterion implies a further branching for the geodesics that have identical edges thus far to be distinct before they reach t, and since the length of the geodesics in this infinite branching is infinite, we have a contradiction. Hence by the subgraph corollary, $f(e_1) < f(e_2)$ in *G*.

Some of the more useful theorems in graph theory state the identity between local (traversal) properties of graphs, such as the g- and g+ inequalities, and global structural properties. Having defined some traversal properties and theorems, we now define some structural properties that will help to bound the types of graphs in which g- and g+ inequality occur.

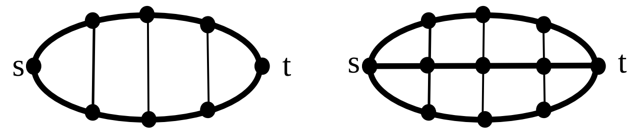
To prove the following theorem, called Subgraph Lemma, we introduce additional terminology. An *interior* of an *s-t* graph *G* is a maximal connected subgraph in $G - s - t$. The *interior C* of a cohesive *s-t* block *B* is $B - s - t$. The lighter solid lines of the *s-t* graphs 3a through 3i in Figure 3 are edges in the interiors of each block.

Subgraph Lemma. If $f(e_1) < f(e_2)$ in an *s-t* subgraph *S* of an *s-t* graph *G* that contains g-edge e_1, ~g edge e_2, and at least one *s-t* geodesic, then $f(e_1) < f(e_2)$ in *G*.

Proof. Since *S* contains an *s-t* geodesic of *G*, say of length k, no addition to *S* of missing nodes and edges from *G* can reduce the *s-t* geodesic length k. When a single path is added to *S*, consisting of missing *u-v* paths in *G* for existing *u,v* nodes in *S*, to form a new *s-t* subgraph *S*' of *G*, for every new *s-t* path in *S*' created through a g-edge e_1, at least two *s-t* paths are created for each ~g edge e_2 in the interior of *S*' connected to e_1. This is because the e_2 edges in this interior can always be traversed in two directions while the e_1 edges can only be traversed in one direction, along a geodesic. Applying additional paths in this way, one at a time, it will always be true that $f(e_1) < f(e_2)$ for each successive subgraph as missing paths from *G* are added. Eventually, *G* will be reconstructed, so that $f(e_1) < f(e_2)$ in *G*.

Subgraph Corollary. It follows as a restatement of the proof of the lemma that once an *s-t* graph fails g-inequality, it can never be restored by the addition of new nodes and edges so long as *s* and *t* and the geodesic length k are constant.

The proof of the next theorem uses two digits (20, 11, 02, summing to two) as a count that describes whether edges in *s-t* graphs connect two g-nodes (20g or 20~g edge), two ~g nodes (02~g nodes), or one of each (a 11~g edge). A ~g edge can connect any combination of g or ~g nodes (20, 11, 02), but a g-edge can only connect two g-nodes (hence 20g-node exists,



(a) g-inequality holds (b) g-inequality does not hold

Figure 4: The *s-t* double and triple switching graphs with geodesic length 4.

Switching Theorem. Every double switching graph, but no triple or higher order switching graph, satisfies the g-inequality where f(e_1) = f(e_2) for every g-edge e_1 and ~g edge e_2.

Proof. We consider a switching graph with geodesics of length r as made up of r-1 switches defined by the ~g edges. There are 2^{r-1} paths through each ~g edge switch, hence f(e_2) = 2^{r-1} and by symmetry f(e_1) = 2^{r-1}. Clearly each ~g edge between the two geodesics creates *s-t* paths that increase f(e_1) and f(e_2) for each g-edge e_1 and for each ~g edge e_2. The value of f(e_1) for every g-edge e_1 will be increased for each node-independent ~g path between the two geodesics.

Graph 3i in Figure 3 shows a triple switching graph in which f(e_1) = 3 < f(e_2) = 4 for each g-edge e_1 and ~g edge e_2, and the disparity is greater for the triple switching graph in Figure 4. For *s-t* paths of length $d \geq r$, let $f_d(e_1)$ and $f_d(e_2)$, respectively, be the number of paths that pass through a g-edge e_1 and through a ~g edge e_2. For paths of length $d = r$, $f_d(e_1) = 1$ and $f_d(e_2) = 0$ for all e_1, e_2. It is easy to verify, given the regular construction of switching graphs, that for paths of length $d > r$ and $m > 2$, $f_d(e_1) \leq f_d(e_2)$ for any e_1, e_2, and if in addition $r > 2$, then $f_d(e_1) < f_d(e_2)$ for any e_1, e_2. Hence, if sum $f_d(e)$ is the sum of $f_d(e_1)$ over all $d \geq r$, then for all higher order switching graphs with $r > 2$ and $m > 2$, sum $f_d(e_1) \geq$ sum $f_d(e_2)$ for any e_1, e_2. Since for any e, f(e) = sum $f_d(e)$ over all $d \geq r$, sum $f_d(e_1) \geq$ sum $f_d(e_2)$ implies f(e_1) $\geq$ f(e_2) and thus the g-inequality holds for all g-edges e_1 and ~g edges e_2.

g-Inequality Observation. An *s-t* graph *G* satisfies g-inequality if and only if every cohesive *s'-t'* block *G'* of *G* satisfies g-inequality, and the same for g+ inequality. Recall that an *s-t* graph *G* is not disconnected by removal of nodes *s,t*. The criterion follows by construction from definitions of *s-t* graph *G* and *s'-t'* block *G'*.

some useful first approximations to understanding our problem.

By construction, every cohesive *s'-t'* block *B* of an *s-t* graph *G* is itself an *s-t* graph, and the *s'-t'* geodesics of a cohesive *s'-t'* block *B* of an *s-t* graph *G* are subpaths of the *s-t* geodesics of *G*. Hence we may write *s-t block* of *G* to mean, without ambiguity, a cohesive *s'-t'* block of an *s-t* graph. For simplicity, we often are concerned only with properties of *s-t* blocks, since their properties as regards edge traversal will generalize to *s-t* graphs.

Two *u-v* paths are *node-independent* if they have common starting node *u* and terminal node *v* and no other nodes in common. The union of two node-independent paths from *u* to *v* constitutes a cycle. In Figure 1, for example, which is formally identical to graph 3b, the two edge sequences 1,4 and 2,5 are node-independent *s-t* paths (with just *s* and *t* in common), and together they constitute a cycle (edge sequence 1,4,5,2 or closed walk *s,u,t,v,s*). All paths and cycles are routes, and a path cannot contain a cycle. The following definition and theorem gives our first intuition about node-independence in *s-t* graphs.

The *double r-switching graph* is the *s-t* graph with two node-independent s-t geodesics of length *r* such that each geodesic is made up of *s*, then *r*-1 successive u_i and v_i nodes in each respective geodesic for i=1 to *r*-1, where each u_i,v_i pair is an edge, and then t. Recall that f(*e*) is defined as the number of paths, not only geodesics, containing *e*. By construction, then, every edge u_i,v_i is a ~g edge. Figure 4 gives an illustration where $r = 4$, and f(e_1) = f(e_2) = 8 for each g-edge e_1 and ~g edge e_2. The *triple r-switching graph*, also exemplified in the figure, is the *s-t* graph with three node-independent geodesics of length *r* such that each geodesic contains *r*-1 successive u_i, v_i and w_i nodes, respectively, for i=1 to *r*-1, where each u_i,v_i and v_i,w_i pair is a ~g edge. Higher order (*p*,*r*) switching graphs with *p* node-independent geodesics are similarly constructed.

The study of frequency of traversal of different types of edges in relation to different properties of graphs can now begin. The *frequency f(e)* of an edge *e* in an *s-t* graph is the number of distinct *s-t* paths containing *e*. In Figure 3, the labeled edges in *s-t* graphs 3a, 3c, 3d and 3g classify g or ~g edges and indicate the number of distinct *s-t* paths that pass through each edge. The label g6 indicates a g-edge contained in six distinct *s-t* paths while ~g5 labels a ~g edge contained in five distinct *s-t* paths. In graph 3a, the minimum frequency of a g-edge is five, while the maximum frequency of a ~g edge is also five. In graph 3c, the minimum frequency of a g-edge is 5, while the maximum frequency of a ~g edge is 8. Some of the graphs in Figure 3 are the blocks of Figure 2.

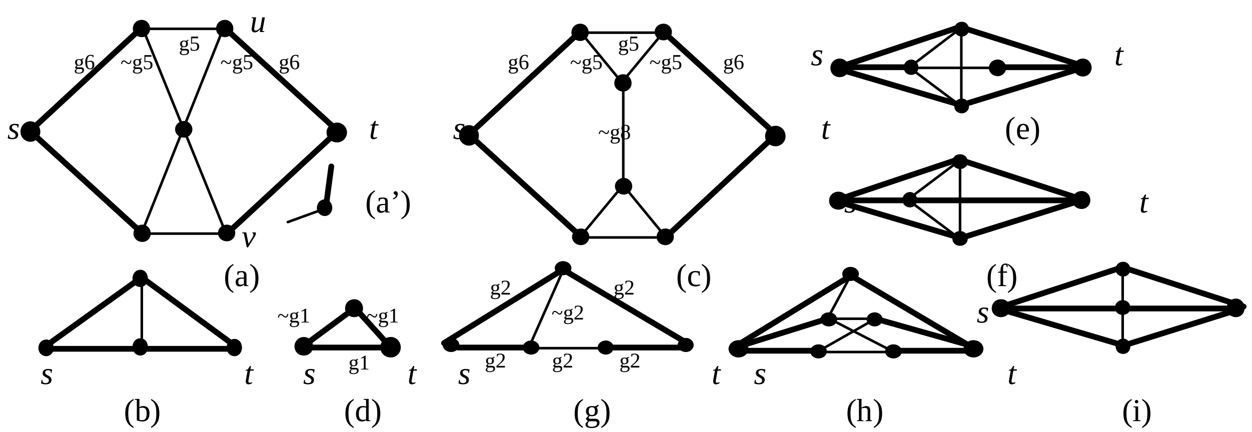


Figure 3: Graphs 3a, 3b, 3d and 3g have g-inequality; 3a also satisfies g+ inequality.

An *s-t* graph *G* satisfies *g-inequality* (short for *the geodetic inequality property*) if for every g-edge e_1 and ~g edge e_2 in *G*, $f(e_1) \geq f(e_2)$. In Figure 3, only graphs 3a, 3b, 3d and 3g have g-inequality. An *s-t* graph *G* satisfies the *g+ inequality* property if it satisfies g-inequality and there is a cutoff value k such that $f(e_3) \leq k$ for every ~g edge e_3, $f(e_4) \geq k$ for every g-edge e_4 and $f(e_4) > k$ for some g-edge e_4. Only graph 3a satisfies g+ inequality, for $k = 5$. For edge traversal, which we will measure probabilistically in our sequel, these two properties give ure 1, since 1, 2, 4, 5 are g-edges, 1,4 and 2,5 are g-paths. Since 3 is a ~g edge, sequences 1,3,5 and 2,3,4 are the ~g paths.

To subdivide our problem into parts, we will define some terms that help to characterize *s-t* graphs by breaking them into natural chunks. The *removal* of a node *u* from a graph *G* results in that subgraph *G – u* of *G* consisting of all nodes of *G* except *u* and all edges not incident with *u* (Harary 1969:11). The removal of a set of nodes is defined by removal of single nodes in succession. A *cutnode u* of a connected graph *G* is a node that when removed from *G*, results in a disconnected graph, *G – u*. A *block B* of a graph *G* is a maximal subgraph of *G* that contains no cutnodes (Harary 1969:26). The graph in Figure 2, for example, has six blocks, some of which are single edges. A *cohesive block C* of *G* (White and Harary 2001) is a block with more than one edge. The *s-t* graph in Figure 2 has four cohesive blocks and two blocks that are single edges.

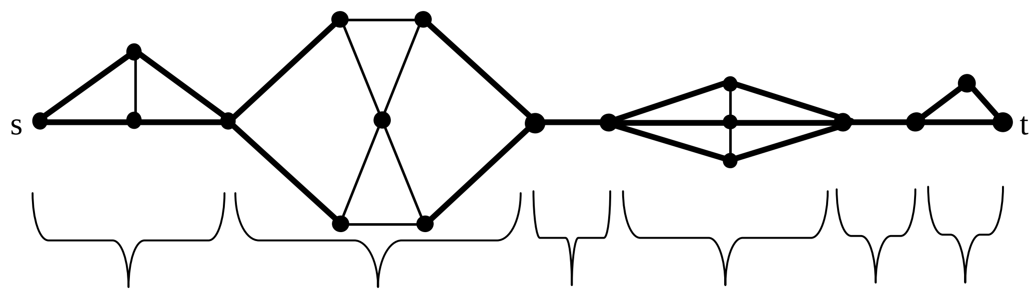


Figure 2: The blocks of an *s-t* graph *G* are maximal subgraphs of *G* that lack cutnodes.

If an *s-t* graph *G* contains cutnodes it can be decomposed into a sequence of pairs of blocks, each contiguous pair having a cutnode of *G* in common; if not *G* is a single block. An *s'-t' block B* is a block of an *s-t* graph *G* with distinguished nodes *s'* and *t'*, which are the respective cutnodes in *B* closest to *s* and *t* (if *s,t* are in *B* then *s'* = *s* and *t'* = *t*, respectively). Since a block lacks cutnodes, every *s'-t'* block *B* of an *s-t* graph *G* is either a single edge or a cohesive block (with deg *s'* and deg *t'* each greater than one).

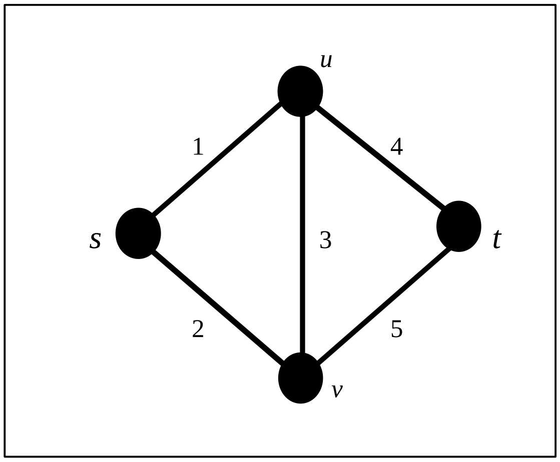


Figure 1: A graph *G* with two distinguished nodes *s,t* and edge**s labeled 1 to 5**

A *2-terminal s-t graph G* (for brevity an *s-t graph*) has two distinguished nodes, *s* standing for source and *t* for terminus, each of which has degree at least 2, and each edge *e* of *G* appears in some *s-t* path. Figure 1 satisfies these conditions for an *s-t* graph, which can be regarded as a maze in which *s* is the entrance, *t* is the exit, and no backtracks are ever necessary.[3] By definition, an operation * on any kind of structure *S* giving *S** is a duality provided *S*** equals *S*. In words, a duality is an operation when applied twice results in the original structure.

***s,t* Duality Principle**. Starting with a 2-terminal *s-t* graph *G*, the graph *G** obtained by interchanging *s* and *t* always results in a 2-terminal graph. Obviously, when this operation is applied twice, the result is the original *s-t* graph *G*. Any of our results for an *s-t* graph will apply to its dual, since the defining properties for an *s-t* graph G apply equally to G*.

In an *s-t* graph, *e* is a *type-g-edge* if it appears in an *s-t* geodesic, otherwise it is *type* **~***g*. In an *s-t* graph, a path is *type ~g* if it contains a ~g edge, otherwise it is *type-g*. For brevity we will henceforth omit the word "type" in referring to ~g edges and g-edges; similarly for paths. In Fig-

[3] Traversal of distinct nodes in a graph thus may represent traversal in a simplified maze in which 'moves' exclude the possibility of a loop, which is an edge joining a node to itself, or of 'deadends,' that is, nodes other than *t* from which, having arrived from s, no further move is possible without backtracking. We will use this construction in our sequel on dynamic maze traversal.

proximate measure of its set of shortest paths and their common shortest path length? To use the theory of graphs, we require some preliminary definitions (Harary 1969).

A *graph G* consists of a set *V* of *nodes* (or vertices) together with a set *E* of edges, each joining a pair of distinct nodes. An edge joining two nodes is said to be *incident* with each of those nodes. The *degree*, deg *u*, of a node *u* in graph *G* is the number of edges incident with *u*. In Figure 1, edges are numbered and nodes are lettered. A *walk* is an alternating sequence of nodes and edges that begins and ends with a node, in which each edge is incident with the two nodes immediately preceding and following it. Thus a walk permits repetition of nodes or of edges. A walk is *closed* if the beginning and ending nodes are the same, and *open* otherwise. A *backtrack* is a node-edge-node-edge-node walk that starts and ends with the same node, traversing the same edge twice, once in each direction. For example, walk *s*,1,*u*,1,*s* in Figure 1 is a backtrack. A *route* is a walk with no backtracks (*route* is a new definition useful for the study of mazes). A *path* is a walk in which all nodes are distinct, and hence the edges are also distinct. The *length* of a *u*-*v* path is the number of edges in it. A *cycle* of length *r* is obtained from a path of length $r \geq 3$ by identifying its first and last nodes. A *u*-*v geodesic* is a shortest *u*-*v* path, and the length of such a geodesic is the *distance* $d(u,v)$ (Harary 1969:13-14). Always starting from node *s* in Figure 1, the sequence 1,1,2,3 of edges is a walk that is not a route because 1,1 indicates a backtrack; 1,3,2,1 is a route that is not a path because edge 1 appears more than once and so do nodes *s* and *u*; 1,3,2 is a cycle; and the paths 1-4 and 2-5 are *s*-*t* geodesics, whereas path 1-3-5 is not one.

search behavior that are crucial to the collective advantages he observes in problem solving. The network structures that we can identify in collective problem-solving examples need to be seen as ‘network externalities’ (Arthur 1990), AW processes of interactions that recursively change the environment of interaction (Lam 1997), or problem-solving contexts that may co-evolve along with the population of organisms (Kauffman 2000). We hold that such processes and contexts are ubiquitous features of evolutionary processes for populations of purposive agents, which includes all living species. The purpose of our graph theoretic exposition are threefold: First, to develop a language in which we can precisely describe the relational structures involved in evolutionary problem of collective advantages that are dependent on network structure. Second, to use this formal language to derive new insights. Third, through this formal logic for the structure of our problem, to show how properties that are emergent, for both the network actors involved and the evolutionary biologists, may be a predictable result of the context in which a co-evolutionary development is occurring.

The graph theoretic problem

Why should marking paths by frequency of traversal lead to identifying shortest paths? An analogous problem in graph theory is: How is it that random path traversals in a network produce more frequent traversal of geodesics than non-geodesics? Note that if such a 'signal' is present in random path traversals, the average of many trials is a more reliable guide to finding shortest paths than is a single trial. The problem is then equivalent to the reliability of measurement, where averaging many less reliable (Moore and Shannon 1956) but independent measures of some construct gives greater reliability than taking just a single measure. The graph theoretic problem that we need to understand in the first instance, however, is this: Under what conditions, if any, can we consider a random route through a maze to yield an apmodel is an abstracted representation of real systems" (Johnson 2000:4). At another level, since the edges of the maze are unweighted (alternate-choice paths at any node are equiprobable at the outset), we can study how the pure graph-theoretic structure of the maze contributes to collective behavior. We do not explore the case of mazes with weighted edges, which as an analytic problem is more complex. The findings presented here can be viewed as an extension in the physical theory of active walkers (AW), in which trail traversal transforms the environment in which walkers interact. [2] Here we provide solutions to the problem of constraints on collective discovery of shortest paths in random path traversal in graphs with designated starting and terminating nodes. The graph theoretical section gives the core definitions leading to the main theorems.

The evolutionary problem

The "trail marker" problem represents a class of processes in which selection need not operate on individuals. Indeed, if there were such selection, it would reduce the variability in the population that provides adaptability for changes in the environment. Nor need selection operate to favor cooperation between individual ants, since no such interactive cooperation need be entailed. Johnson (2000) argues that his simulations, embedded in the type case of networks that represent the structural constraints on collective problem solving, represent non-selective and emergent processes in collective problem solving. We show that it is the network structure itself that gives substance to his type cases, and that determines the experimental results of his simulations. This does not reduce the significance of his findings, nor the fact that it is diversity in the population and the independence of each by agen*t*'s traversal in its

[2] AW theory has been proposed by Lam (1997) as one of the principal processes in nature governing pattern formation, self-organization, and the dynamics of complex systems. Although we provide here a discrete structure application of AW theory, our sequel on dynamics and simulations will introduce methods of random traversal that yield collective solution to the shortest path problem, and gives additional observations about properties of graphs with collective geodesics.

I. Self-organization in collective problem solving

Organizations that preselect clever learners to solve their problems learn quickly to *exploit* their environments but they do so at the expense of *exploration*, which often yields better results, especially in the longer run. So goes March's (1991) conclusion from his simulation of learning in organizations, in which collections of diverse learners frequently outperform collections of learners preselected for their abilities. Similar results are found in simulations by Johnson (2000, 2001), who modeled the ability of learners acting independently to solve problems such as running mazes. Real world examples abound: In the insect world, for example, ants have the ability to find shortest paths given multiple routes between nest and food sources. Early in the process of finding a food source, individual ants tend to explore routes randomly, but lay down pheromones on the paths they traverse. While pheromones decay with time, their residuals accumulate to mark routes more frequently traversed. When food is found, almost all the food carriers make the transition to following the heaviest pheromone trail to the source. The surprise is that this trail is usually a shortest path. The pheromone markers are a way that collective behavior can be coordinated in the absence of direct interindividual cooperation and without competitive selection for individuals with better pathfinding ability. Does this example (Johnson 2000) represent a more general means of self-organization in collective problem solving that operates independently of natural selection on individuals in a population?

The maze traversal model of problem solving, at one level, serves as a representation of a much more general class of complex processes: "a problem that has many decision points and many possible solutions.... All evolutionary systems are sequential in nature (every action of an individual has a prior, different action leading to the present state), and [the maze]

Collective Geodesics and Co-evolution: A Graph Theoretic Structural Model [1]

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Revised version for ***Advances in Complex Systems***, May-2002, Peter Stadler, editor

Abstract. Contributions to complexity theory in relation to social organization are given by original proofs in graph theory that show the structural conditions that maximize the probability of finding shortest-step solutions to problem-solving from start to goal in a network through a set of random paths. The proofs link findings in simulation models, such as James March's (1991) discovery of advantages to exploratory (random) behaviors over selection for exploitation in human problem solving, to features of "dual" social organization that enhance these advantages. Examples are the moiety systems frequently found in pre-state kinship-based societies, structured competition, and certain collaborative disciplines in business organization. Comparable applications are found in the problem of how ants' random traversal of the spatial maze of their environment to find food sources structure a cooperative solution to the minimum search problem through the structure of their pheromone marking of paths, which limits the interaction space to one that resembles a spatially localized moiety-like structure of traversal. Hence we show, in the general case, the conditions under which certain very general classes of mazes (as appropriately structured start-to-target graphs) allow shortest paths to be found by aggregating certain types of random individual behavior. The "collective intelligence" of these aggregate solutions to problem solving, then, reside as much in the structure of the maze or graph as in the ability to record information collectively. The argument supports the importance of co-evolution between species-actants and species-environments.

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[1] We thank Norman Johnson, Joel Gunn and an anonymous referee for helpful critiques and commentary. This research was supported by NSF grant #BCS-9978282, "Longitudinal Network Studies and Predictive Cohesion Theory," to D. R. White, 1999-2001, with consultant Frank Harary.
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