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Networks embedded in *n*-dimensional space: The impact of dimensionality change

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Abstract

Social networks can be embedded in an *n*-dimensional space, where the dimensions may reveal or denote underlying properties of interest. When the pertaining actors occupy niches of resources in this space, e.g., organizational niches of affiliates, we show there exists a non-monotonic effect of dimensionality change. Depending on niche width, relatively narrow or wide, dimensionality change has opposing effects on niche volume.

JEL classification: C60; L0; L1

Keywords: Network embedding; Hypersphere volume; Niche

Social networks can be embedded in an *n*-dimensional space (Freeman, 1983), where the dimensions may reveal or denote underlying properties of interest. The most straightforward and most often used embedding is into a Euclidean space, as is done in data exploration techniques like multidimensional scaling (Cox and Cox, 1994). For voluntary organizations the dimensions may stand for socio-demographic characteristics of their members (McPherson, 1983). For political parties, in turn, the dimensions may be issues on which voters have an opinion (Bueno de Mesquita and Stokman, 1994; Downs, 1957). McPherson (2004) proposed to speak in general about Blau-spaces when the dimensions stand for sociological properties, as a tribute to Peter Blau. In this research note we show that dimensionality change of a Blau-space, for instance the emergence of a new issue on the political agenda or the disappearance of an old issue, may result in counter-intuitive effects on the viability of the actors involved: space dimensionality change can have an opposing impact on the volumes of broad and of narrow niches in Blau-space.

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Social organizations, as well as other social actors, have limited capacity to acquire resources from their environment. They face trade-offs in deploying their given adaptive capacity (Hannan and Freeman, 1977; Hannan et al., 2003), which in biology is called the *principle of allocation* (Levins, 1968): actors can be moderately proficient at collecting resources – e.g., attracting affiliates such as customers, voters, or members – from a wide niche in Blau-space, or they can be highly proficient at reaping a narrow niche. Actors with a wide niche are called generalists whereas actors with a narrow niche are called specialists. Actual fitness as well as realized niches also depend on competition, which we do not take into account in this paper because it confounds the effect on which we focus here; in the remainder we discuss fundamental niches. In competitive settings, the niche volume effect pinpointed here simply adds to niche overlap effects.

For some organizations, in particular voluntary organizations and political parties, the principle of allocation is reinforced by the requirement of having social cohesion among affiliates. According to the *homophily principle* (McPherson et al., 2001), social cohesion is stronger among more similar actors. Thus, if an organization attempts to appeal to more diverse potential affiliates (wide niche), the lower their social cohesion will be; the fundamental niche increases, but becomes (too) sparsely occupied, so there may even be an upper bound on niche span. In contrast, an organization can have a narrow niche with a potentially high cohesion among its affiliates; this organization has a small number of affiliates in a possibly more densely filled niche. For example, the appeal of Bristol Cars seems to be limited to British aristocracy,[2] with very expensive hand made cars that are for sale only in a single shop in London. This narrow niche company turns out to be a long time survivor even though its niche volume is small. Which niche strategy is more viable is determined by long run patterns of environmental change.

To see how and why dimensional change affects niches, we start out with two model assumptions. First, resources (affiliates) are by and large homogeneously distributed *within* the niche, while the overall resource distribution might be inhomogeneous, for example, a Gaussian. This is a weaker assumption than stipulating a homogeneous resource distribution for the entire Blau-space. If the distribution is far from being homogeneous even within the niche, then the inhomogeneity-based effects add up to the volume-related dimension effects discussed in this note.

Second, niches are assumed to be spherical with radius r. To this end, we assume standardized Blau-space variables with unity weights. Weights, standing for the importance of variables, involve affine transformations that lengthen or shorten objects along axes.[3] Affine transformations change distances, and so object volumes, in the same proportion along the whole space; therefore, volume ratios of different niches under comparison are left intact. Since we will investigate these changes in niche volumes by volume ratio changes, the argument can proceed with unit weights. With these provisions in place, the shape of a fundamental niche is approximated by an n-dimensional hypersphere (Péli and Nooteboom, 1999), provided that distance is Euclidean. With certain association measures in place, e.g., if the distance between the focal organization and the potential affiliate cannot surpass a threshold along any dimension, niches can be rectangular (boxicity, Freeman, 1983).[4] The effect to be demonstrated here relies on sphere properties, so we continue with Euclidean distance-based association, and so, with spherical niches.

We will compare two social actors, for instance organizations, with different values of niche radius r, a generalist and a specialist. Now let there be an externally induced change of

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[2] This holds at least before they launched the Fighter model (www.bristolcars.co.uk/index2.htm).

[3] http://mathworld.wolfram.com/AffineTransformation.html.

[4] The pertaining association measure between vectors (actors) $\boldsymbol{a}$ and $\boldsymbol{b}$ is: $A_{\mathrm{ab}} = \max_{i=1...n}|a_i - b_i|$. Affiliation is only possible if $A_{\mathrm{ab}} \leq \delta$, where δ is a fixed value.

Table 1
The values of γ_n up to $n = 10$

n	1	2	3	4	5	6	7	8	9	10
γ_n	2	π	$\frac{4}{3}\pi$	$\frac{1}{2}\pi^2$	$\frac{8}{15}\pi^2$	$\frac{1}{6}\pi^3$	$\frac{16}{105}\pi^3$	$\frac{1}{24}\pi^4$	$\frac{32}{945}\pi^4$	$\frac{1}{120}\pi^5$

Source: http://mathworld.wolfram.com/Hypersphere.html

dimensionality of the Blau-space. This can happen for political parties when a new important issue enters their agenda and adds a dimension, or when political discourse reduces to a few or even to a single issue, like it happens in times of wars and under terrorist attack (e.g., “Are you with us or against us?”). How could the number of affiliates be affected? If dimensionality decreases while keeping r constant, there is an increasing density, ρ_n, of potential affiliates in the populated segment of the Blau-space, while their total number stays the same because they are concentrated in a lower dimensional hypervolume, with less unitary cells so to speak. Vice versa, if dimensionality increases, there is a thinning out of affiliates because they are spread out over more dimensions. Furthermore, the appeal of an organization to potential affiliates, $A(r)$, is a monotonically decreasing function of r, due to the principle of allocation, possibly reinforced by (lack of) social cohesion. Last but not least, the volume of the niche, V_n, plays a role, as elaborated below. Actual appeal (Hannan et al., 2003) can be operationalized as the proportion of people in the niche that decide to affiliate with the organization when its niche radius is r. The number of affiliates within a niche of radius r embedded into an n-space, $S_{r,n}$, can be written as

$$S_{r,n} = V_n \rho_n A(r) \tag{1}$$

Since only the number of Blau-space dimensions changes, not the niche span, appeal, $A(r)$, stays the same. Density does increase or decrease, but does so for all organizations in the same proportion, hence, it is irrelevant for our comparison. So, changes of numbers of affiliates ultimately rest on changes of the hypervolume of the niche. The interesting point is that dimensionality change has a *non-monotonic* effect on this hypervolume. The hypervolume of an n-sphere is

$$V_n = \gamma_n r^n \tag{2}$$

where γ_n depends only on n. The volumes for the range $n = 1$–10 dimensions, which encompasses most social science applications, are given in Table 1.

When $r = 1$, the maximum volume is at five dimensions. However, for higher or lower values of r, this is no longer true, and the maximum moves up or down with the number of dimensions, respectively (see Fig. 1).

Fig. 2 displays the changing volume maxima as a function of sphere radius. Note that the non-monotonic volume effect is restricted to a certain parameter range. The graph also reveals that the peak disappears at lower radii; the hypersphere volume decreases monotonically with dimensionality if $r \leq 2/\pi$ (see the proof in Appendix A).

Now, let a generalist have $r = 1$, and as pointed out in Fig. 1, its maximum hypervolume is at $n = 5$. Let a specialist have $r = 0.7$; its maximum volume is at $n = 2$. Fig. 2 shows that dimension change has an opposite impact on the respective niche volumes if the shift takes place along the [2; 5] range. So, if n changes, say, from 4 to 3, the volume of the specialist niche *increases*, while it *decreases* for the generalist (see the formalized general proposition in Appendix B).

This implies that only by dimensionality change of the Blau-space, one organization can gain affiliates while another organization loses. This effect may explain why elections at times of

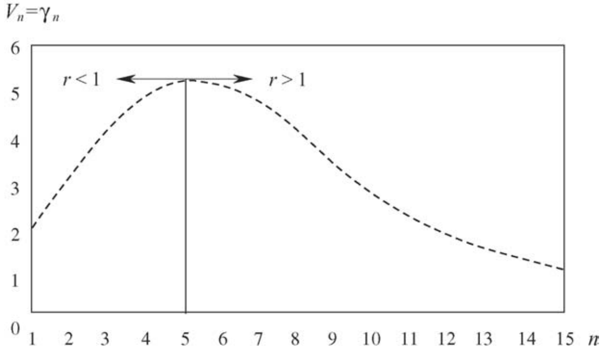

Fig. 1. Unit sphere ($r = 1$) volume V_n in different dimensions (cf. Table 1). Arrows show the direction of maximum shift with radius.

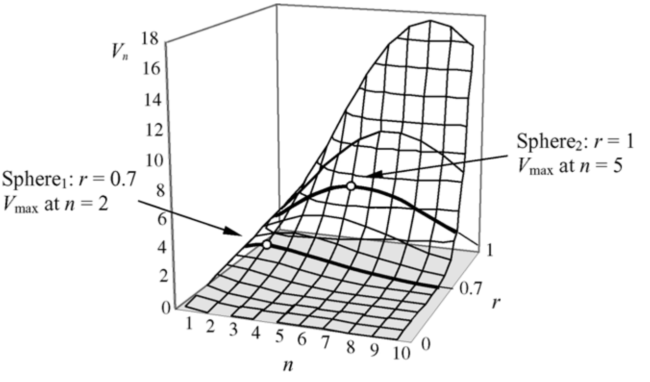

Fig. 2. Sphere volume V_n change with radius r and dimension n.

political turmoil, or emerging and fading market fashions, sometimes lead to quite unexpected outcomes. When the dimensionality of a political space decreases, a "specialist" with a more clear profile may win. Conventional wisdom has it that in times of political crisis, the appeal of a party with a simple and clear profile increases, and more people in its fundamental niche become affiliates (voters in this case). As we have just shown, though, this party may also win without any change of its appeal, just by the increasing volume of its niche. Of course the latter phenomenon does not exclude the former, and in our example both effects work in the same direction.

Appendix A

Proposition 1. Hypersphere volume V_n has a maximum along the [1; 10] dimension range if:

$$\frac{\pi}{2} < r < \frac{4.063}{\pi} \tag{3}$$

Proof. V_n has a maximum along the [1; 10] range if it is increasing between $n = 1$ and 2 and if it is decreasing between $n = 9$ and 10. All radii r for which moving from $n = 1$ to 2 increases hypersphere volume V_n satisfy (4), cf. Table 1:

$$2r < r^2\pi \tag{4}$$

$$r > \frac{\pi}{2} \tag{5}$$

All r for which moving from $n = 9$ to 10 decreases hypersphere volume V_n satisfy (6), cf. Table 1:

$$r^9 \frac{32}{945}\pi^4 > r^{10}\frac{1}{120}\pi^5 \tag{6}$$

$$r < \frac{4.063}{\pi} \tag{7}$$

Since γ_n has a single extreme point and r^n is monotonic in n, $V_n = r^n\gamma_n$ can have only a single maximum. □

Appendix B

Proposition 2. Let R and r be the radii of two spherical niches embedded into a Blau-space, and $R > r$, $\frac{\pi}{2} < R < \frac{4.063}{\pi}$. Then there exists an $[n_1; n_2]$ range along which increasing space dimension changes the ratio of affiliates in the two niches in favor of the wider niche (and vice versa).

$$\frac{S_{R,n+1}}{S_{r,n+1}} > \frac{S_{R,n}}{S_{r,n}} \quad n_1 \le n < n_2 \tag{8}$$

Proof. Instantiate (1) to (8). Simplifying the obtaining expression gives (9) where $V_{r,n}$ is the n-dimensional hypersphere volume at radius r:

$$\frac{V_{R,n+1}}{V_{r,n+1}} > \frac{V_{R,n}}{V_{r,n}} \quad n_1 \le n < n_2 \tag{9}$$

Let n_1 and n_2 be, respectively, the volume maximizing dimensions at radii r and R in place. Then, inequality (9) holds true, see Fig. 2. Thus (8) also holds true. □

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