Complexity Theory and Models for Social Networks

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Much work in complexity theory employs agent-based models in simulations of systems of multiple agents. Agent interaction follows some standard types of network topologies. My aim is to assess how recent advances in the statistical modeling of social networks may contribute to agent-based modeling traditions, specifically, by providing structural characterizations of a variety of network topologies. I illustrate the points by reference to a computational model for the evolution of cooperation among agents embedded in neighborhoods and by reference to complex, real social networks defined by the ties of political support between US Senators as revealed through ties of cosponsorship of legislation. © 2003 Wiley Periodicals, Inc.

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The purpose of this article is to contribute to the greater understanding of network topologies for complexity analyses, particularly, analyses that deploy agent-based modeling strategies. In such models, the pattern of interactions between agents is crucial, and the network topology that emerges central, to the aggregate outcomes emergent from the local interactions of agents. Standard interaction protocols produce highly stylized network topologies. My short run aim is to analyze these topologies using recent advances in the statistical modeling of social network, a set of advances collected under the generic name of "exponential random graph" (erg) models. This task will serve to introduce these advances, the assumptions upon which they are based, and the analyses they make possible. My long-run aim is to demonstrate how these erg models could be used to provide more sophisticated and realistic network topologies for complexity analyses.

In the following sections, I review formal models for social networks, both theoretical and methodological. I then discuss the types of network topologies found in agent-based models and illustrate the issue with a particular example, the network topology underlying the work of Macy and Skvoretz [1], who produced a system in which it was possible to evolve cooperation in the one-shot Prisoner's Dilemma (PD) between strangers, a result, however, that was highly dependent on parameters of the basic network topology. I then analyze this topology using a very simple erg model with the aim of providing a statistical understanding of the difference between topologies in which the interaction basis is sufficient to allow stranger cooperation to emerge and topologies in which it is not sufficient. I then compare these statistical characterizations to a "real" network that maps the ties of political support among US

Senators as revealed in their cosponsorship of one another's legislation. On one hand, the question is: is cooperation likely among Senators who are relative strangers—that is, does the statistical characterization of the Senate match the statistical characterization of a network topology that would be sufficient to support such cooperation. On the other hand, the question is what novel interaction protocols could be proposed that would yield a network topology whose statistical profile matches that found in the legislative body—a question of broadening the available network topologies for complexity analyses.

MODELS FOR SOCIAL NETWORKS

Models for social networks may be classified by their origins [2]. If the origins lie in general methodological techniques for data representation, the models may be called methodological models. If the origins lie in formal theoretical analysis of specific social forces shaping contact or connection patterns, the models may be termed theoretical models. Both types of models perform the same function of relating theoretical concerns to relevant data. However, in much social science, the models are methodological because most theories in social science are not formally stated. If a theory is not formally expressed, models for data patterns that embody its logic cannot be derived. The only alternative uses general methodological techniques to represent data patterns and then "interprets" such models' effect parameters in terms of the theory's constructs.

In the context of models for social networks, the situation is atypical for social science in that theoretical models came first and then methodological models. In fact, the most sophisticated models today are "exponential random graph models," a form of general methodological model, and, with one major exception, the development of theoretical models has been retarded by their analytical complexity. This analytical complexity is side-stepped by erg models. A brief review of early theoretical developments and then a review of the development of statistical-methodological models follows.

Random and biased net theory was the earliest attempt to formally model social (and other networks). Rapoport introduced it in a series of articles [3–7]. Rapoport and colleagues used the framework to model aspects of friendship networks in two junior high schools [8,9]. Fararo and Sunshine [10] made significant theoretical extensions in their study of a large friendship network also among junior high school students.

In biased net theory, a network is the outcome of a stochastic process that has random and biased elements. Aggregate patterns in network structure emerge from local events of connection, that is, complexity at the aggregate level arises from the compounding of relatively simple, local events of connection. However, the stochastic nature of the model makes analytical derivations impossible and exploration of a model's consequences often relies on approximation assumptions.

In Rapaport's original presentation of biased net theory, biases were defined with the aim of deriving the connectivity of the network as the limit of a procedure of tracing outward from a small, randomly selected set of nodes to the rest of the population. The connectivity of the net was defined by recursion formula that expressed the fraction of new nodes reached at distance $t + 1$, $P(t + 1)$, from the start nodes as a function of the proportion of nodes already reached at distance t or less, $X(t)$, and of structural features of the network. One important feature was the contact density of the network, denoted by a and defined as the constant number of "axons" or contacts emitted by each node. In a random net, the tracing formula could be expressed as follows:

$$P(t + 1) = (1 - X(t)(1 - e^{-aP(t)})$$

In a biased net, other structural features, called biases, changed the tracing formula. Two of these important biases were reciprocity or "parent" bias and the closure or "sibling" bias, defined as follows:

$$\pi = \Pr(x \rightarrow y | y \rightarrow x)$$

$$\sigma = \Pr(x \rightarrow y | \exists z \ni z \rightarrow x \wedge z \rightarrow y)$$

That is, the reciprocity bias suggests that the probability of a tie from x to y is elevated if there is a tie from y to x. The sibling bias suggests that the probability of a tie from x to y is elevated if there is a third node z that has a contact to x and a contact to y. In this case the tracing formula has the following form:

$$P(t + 1) = (1 - X(t)(1 - e^{-\alpha P(t)})$$

where for t = 0, $\alpha = a$ and for $t > 0$, $\alpha = a - \sigma(a - 1) - \pi$. Note that the biases (probabilistically) reduce the number of contacts available to be sent to nodes as yet unreached in the tracing process. Note also that the conceptualization of biases "conserves density." That is, the biased net in which each node contacts a others is just as dense as the random net in which each node contacts a others.

Until Fararo and Sunshine, biases were "structural," that is, biases that enhanced the probability of a tie's occurrence given the structure of ties that surround a particular pair of actors. The reciprocity or mutuality bias is a simple example—the parameter captures the idea that a tie from x to y is more likely than chance if there already is a tie from y to x. Fararo and Sunshine introduced "compositional" biases: biases that impact the location of a tie, depending on the similarity or difference in actor attributes, such as, their status as delinquents. Later research used the specific compositional bias they introduced—the inbreeding bias—as the foundation of a formalization of Blau's influential macrosociological theory of social structure [11]. Fararo and Skvoretz [12–16] developed this formalization in a series of articles. These articles introduced an additional compositional bias—an outbreeding bias—necessary to model ties such as marriage in relation to the compositional dimension of gender. The articles also provided formal models for situations in which multiple dimensions are in play simultaneously and for situations in which the compositional dimensions are ranked dimensions, such as education and age, Blau's graduated parameters. Additional research based on these articles used biased net concepts to formalize Granovetter's [17] strength of weak ties arguments [18] and then to unify these arguments with Blau's macrosociological theory in a formal synthesis.

Research into the foundations of biased net theory [19,20] proposed Monte Carlo simulation methods to generate networks of specific size subject to specified levels of various bias factors. The research also proposed a way of estimating bias parameters using a cross-classification of choice patterns in dyads by the number of co-nominations received by the dyad. Both efforts were not entirely successful and furthermore, cast doubt on the validity of certain approximation arguments traditionally used in biased net theory to derive important network properties of interest, such as connectivity. Development of the approach, therefore, stalled.

On the other hand, methodological models for social networks began with early tests for departure from randomness searching for reciprocity effects in dyads or for transitivity effects in triads [21,22]. The general methodology of categorical data analysis represented by log-linear models for cell counts were adapted by Holland and Leinhardt [23] to social network data in the form of directed graphs, calling the model the "p_1" model.

This model assumes dyadic independence: pattern of arcs in *ij* pair is independent of the pattern of arcs in any other pair including ones containing *i* or *j*. In the basic model, the effects taken into account on the probability of a "1" in the *ij* cell include the following: density (θ), the differential expansiveness of nodes (α_i), the differential attractiveness of nodes (β_i), and reciprocity (ρ). The basic equations for the model are

$$\Pr(X_{ij} = 1 \;\&\; X_{ji} = 1) = \frac{e^{2\theta + \alpha_i + \alpha_j + \beta_i + \beta_j + \rho}}{1 + e^{\theta + \alpha_i + \beta_j} + e^{\theta + \alpha_j + \beta_i} + e^{2\theta + \alpha_i + \alpha_j + \beta_i + \beta_j + \rho}}$$

$$\Pr(X_{ij} = 1 \;\&\; X_{ji} = 0) = \frac{e^{\theta} + \alpha_i + \beta_j}{1 + e^{\theta + \alpha_i + \beta_j} + e^{\theta + \alpha_j + \beta_i} + e^{2\theta + \alpha_i + \alpha_j + \beta_i + \beta_j + \rho}}$$

$$\Pr(X_{ij} = 0 \;\&\; X_{ji} = 1) = \frac{e^{\theta + \alpha_j + \beta_i}}{1 + e^{\theta + \alpha_i + \beta_j} + e^{\theta + \alpha_j + \beta_i} + e^{2\theta + \alpha_i + \alpha_j + \beta_i + \beta_j + \rho}}$$

$$\Pr(X_{ij} = 0 \;\&\; X_{ji} = 0) = \frac{1}{1 + e^{\theta + \alpha_i + \beta_j} + e^{\theta + \alpha_j + \beta_i} + e^{2\theta + \alpha_i + \alpha_j + \beta_i + \beta_j + \rho}}$$

The usefulness of this model is limited by independence assumption. It is widely recognized that this assumption clearly oversimplifies matters. As one example of the inappropriateness of the dyadic independence assumption, triadic effects such as the presence of an *ij* tie being significantly more likely if there are several others *k* who have ties to *i* and to *j* abound in real social networks. Modeling these effects is beyond the capability of statistical models that assume dyadic independence. The new statistical approaches, the erg family of models, explicitly model nonindependence among dyads by including parameters for structural features that capture hypothesized dependencies among ties [24–29].

One common way to think about erg models is that they express the probability of a digraph G as a log-linear function of a vector of parameters θ, an associated vector of digraph statistics $x(G)$, and a normalizing constant $Z(\theta)$:

$$P(G) = \frac{\exp(\theta x(G))}{Z(\theta)}$$

The normalizing constant insures that the probabilities sum to unity over all digraphs. The θ parameters express how various properties of the digraph affect the probability of a specific digraph's occurrence. For instance, the amount of mutuality, i.e., reciprocated choices in the graph, is such a property. Conceptually, if there is a "strong" force toward mutuality, then digraphs with many dyads in which choices are reciprocated are more probable (net of other factors) than digraphs with few dyads in which choices are reciprocated. The vector of parameters, θ, captures the impact of different properties on digraph probability. If a parameter estimate for a specific property is large and positive, then graphs with that property have large probabilities. If a property has a negative coefficient, then graphs with that property have small probabilities.

Because of the analytical intractability of the normalizing constant, the above form of the model cannot be directly estimated. Rather an indirect estimation procedure is proposed that focuses on the conditional logit, the log of the probability that a tie exists between *i* and *j* divided by the probability that it does not, given the rest of the graph [24,30]. Derivation of this conditional logit shows it to be an indirect function of the explanatory properties of the digraph. Specifically, it is a function of the difference in the values of these variables when the tie between *i* and *j* is present versus when it is absent, as specified in the following equation:

$$\text{logit}[P(x_{ij} = 1|G^{-ij})] = \theta[x(G^{+}) - x(G^{-})]$$

where x_{ij} is the tie from i to i, G^{-ij} is the digraph including all adjacencies except the i,jth one, G^{+} is G^{-ij} with $x_{ij} = 1$, whereas G^{-} is G^{-ij} with $x_{ij} = 0$.

In the logit form of the model, the parameter estimates have slightly different interpretations. For instance, if the mutuality property has a negative coefficient, then in the exponential form, we may say that a graph with many mutual dyads has a lower probability than a graph with few mutual dyads. In the logit form, the interpretation is that the log odds on the presence of a tie between i and j declines with an increase in the number of mutual dyads that would be created by its presence. The importance of the logit version of the model lies in the fact that, as Strauss and Ikeda [30] show, the logit version can be estimated, albeit approximately, using logistic regression routines in standard statistical packages. The estimation approach is called "pseudolikelihood" and is widely used in spatial modeling where similar equations with intractable normalizing constants are encountered. Underlying this model is the assumption that the logits of the conditional probabilities are statistically independent [25, p. 48].

Digraph properties that may be used as independent variables can include dyad effects such as mutuality, triad effects such as transitivity or closure, and even higher order effects such as the closure of four actor subgraphs into generalized exchange structures or even, as suggested by Wasserman and Pattison [24], the overall degree of the digraph's centralization. In any case, two common assumptions are made to limit parameters that must be estimated. The first assumption is a homogeneity assumption, that is, that a particular effect does not depend on the identities of the nodes involved. So, for instance, it is assumed that the mutuality effect does not depend on which dyad is under consideration. A second common assumption is called the "Markov" assumption. The idea here is that edges or arcs can be conditionally dependent only if they share at least one node in the original graph. Such an assumption would rule out using properties of four node subgraphs or overall graph properties like centralization as independent variables predicting the log odds that a tie is present versus absent. In the illustrations we consider in the next section, we adopt this Markovian assumption. Further simplification results from modeling graphs rather than digraphs. For examples of more complicated analyses of digraphs in the context of comparing networks, see Faust and Skvoretz [31] and Skvoretz and Faust [32].

AGENT-BASED MODELS: STATISTICAL ANALYSIS OF AN ILLUSTRATIVE NETWORK TOPOLOGY

Agent-based models have a number of common features. The focus of attention is on systems consisting of multiple agents, and the concern is with the emergence of system regularities from local interactions between agents. Agents have internal states and behavioral rules, and the rules may be fixed or changeable through experience and interaction. Agents are boundedly rational; they have only limited information processing and computational capacity. Agents interact in an environment that provides resources for their actions. Typically, agents and/or the rules they use thrive or die based on their success in obtaining resources. Agent-based models are the paramount tools of what Epstein and Axtell [33] call "generative social science," social science whose overarching issue is to explore what micro-specifications of agents and their interaction protocols are sufficient to generate macro-phenomenon of interest.

To make these points concrete, we consider a specific example of such a model introduced by Macy and Skvoretz [1] in their computational study of cooperation between strangers in the one-shot PD game. The details of their model are as follows. The system consists of 1000 agents. Each agent's behavior is coded by a bit string 15 tokens long. The string identifies actions the agent may take (play or not and if play, cooperate or defect) based on information about self and a potential partner. Each string embodies a set of rules by which the information is used to produce (or not) certain actions. An agent's rule set is not constant but may change by imitation of more fit partners. The environmental resources consist of the payoffs from the PD and from the decision to play or not the PD with a particular partner.

For present purposes our concern is the network topology that arises as partners are selected for potential play. Common topologies are as follows: fully random selection, in which any other agent is equally likely to be chosen as a partner, spatially defined selection in which agents occupy cells on a grid and select from their immediately adjacent neighbors (either in their Moore or von Neumann neighborhood), and stratified random selection, in which the choice of partner is random but only from a subset of the overall population, an agent's social neighborhood. The model for cooperation uses the third topology, which is, itself, varied in two ways: first, by size of neighborhood from small (n = 10) to relatively large (n = 50), and second by "embeddedness," as defined by a bias probability that makes it more likely that a neighbor (someone in the agent's social neighborhood) versus a stranger (someone in any other social neighborhood) is selected for potential play. Embeddedness corresponds to an inbreeding bias event in biased net theory—that is, it is the probability of an event such that if it occurs, the partner is selected from the agent's neighborhood with probability 1, and if it does not occur, then choice is made at random from the entire population (and hence may result by accident in the selection of a neighbor). (The relative payoffs from refusal to play and from the PD outcomes are also varied and do affect the emergence of cooperation between strangers but this outcome is of no interest in the present context.)

The tables and analyses that follow are based on simulation runs of 500 generations, with each generation involve 10,000 pairwise "encounters." Each encounter may potentially lead to a play of the PD game if both partners choose to play. If one or both partners refuse play, the exit payoff of 1 is earned. The PD payoffs are $T = 4$, $R = 3$, $P = 1$, and $S = 0$. Whether play occurs or not, the less fit partner may elect to "imitate" the more fit partner. Imitation is implemented by the less fit partner changing each site on the bit string defining its strategy to the value at that site on the more fit partner. Such a switch occurs with probability 0.5. There is also some small probability of a mutation (0.01) that changes the value at a site to its complement. A number of statistics on play and cooperation are calculated each generation. The tables that follow are based on the last, 500th generation.

The first table compares different values of the network topology parameters and the prevalence of cooperation between strangers. It is clear from the table that the topology matters: small neighborhoods and high levels of embeddedness are conducive to relatively widespread cooperation between strangers. When the population is divided into 100 neighborhoods of size 10 and embeddedness is 0.90 (so a neighbor is selected 90% of the time for sure and 0.1% of the time by chance), the level of cooperation between strangers, when such encounters occur, is 60%. When the population is divided into neighborhoods of size 50 and embeddedness is 0.50 (so a neighbor is selected 50% of the time for sure and 2.5% of the time by chance), the level of cooperation between strangers is quite low at 5%. Note that in the first case, only 9.9% of all encounters are with strangers whereas in the second case 47.5% are. The third example in Table 1 has large neighborhoods but a high level of embeddedness. In this case 9.5% of all encounters are with strangers and a neighbor is selected 90% of the time for sure and 0.5% of the time by chance. The level of cooperation between strangers is 45%.

The networks created by these ties are diagrammed in Figures 1, 2, and 3. (In these figures, nodes that belong to different neighborhoods are differently colored. Size differences are further used to differentiate between neighborhoods. The graphs are drawn using Batagelj and Mrvar's [34] Pajek using its 3D Fruchterman Reingold algorithm.) Overall the first network is the least dense at 0.0097. The two networks with neighborhoods of size 50 are about twice as dense, namely, 0.017 when embeddedness is high and 0.019 when it is low. The maximum density possible would occur when every encounter was between a different pair of agents. That maximum density is 0.020. Therefore, clearly one effect of high embeddedness is a pair of agents can participate in multiple encounters. Table 2 shows the distribution of encounters per pair in these networks. The effect of high embeddedness is clearly visible.

TABLE 1

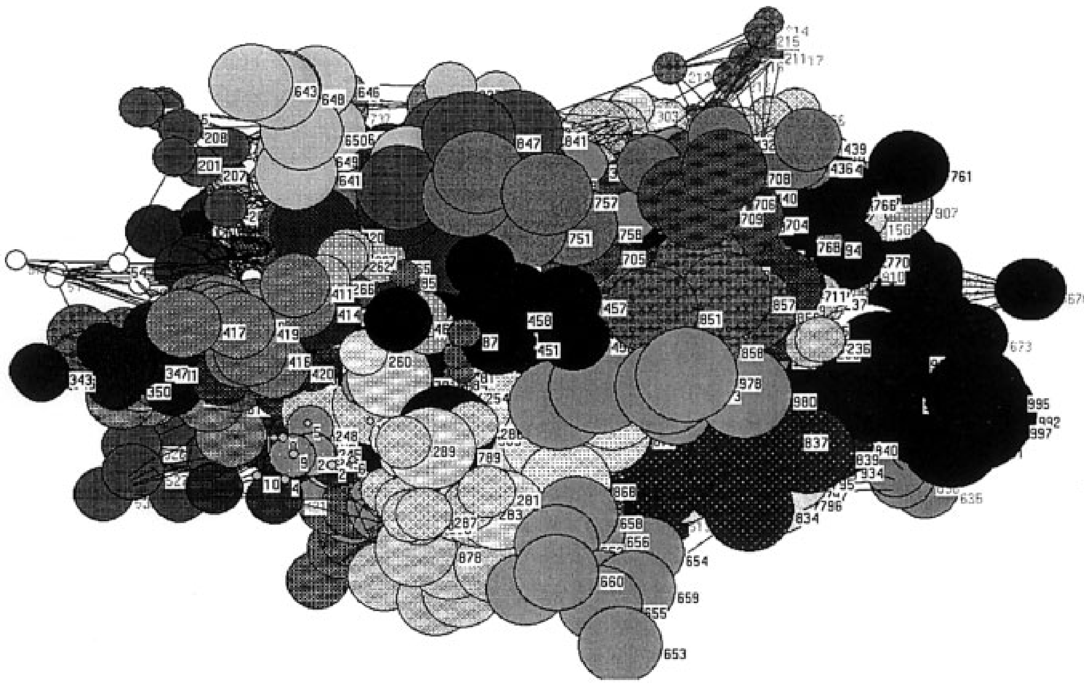
Network Topology and Cooperation between Strangers

Neighborhood Size	Embeddedness	Level of Cooperation between Strangers
10	0.90	0.60
50	0.90	0.45
50	0.50	0.05

A final point to observe is the amount of connectivity between neighborhoods because this feature of the network is related to the potential for cooperation between strangers to evolve and withstand assault. Table 3 shows how these networks differ for various cutoff values. For instance, if we count two neighborhoods as connected if at least one pair of agents have encountered each other, then in the first case of 100 neighborhoods of size 10, only 18% of the neighborhoods are connected to each other. If we compare the two cases in which there are 20 neighborhoods of size 50, it is very clear that embeddedness serves to disrupt direct links between neighborhoods. For instance, if we require that 10 of the potential 2500 agent to agent links be present to count the neighborhoods as connected, every neighborhood is directly connected to every other neighborhood when embeddedness is 0.50, but only just under 5% of the neighborhoods are directly connected when embeddedness is 0.90.

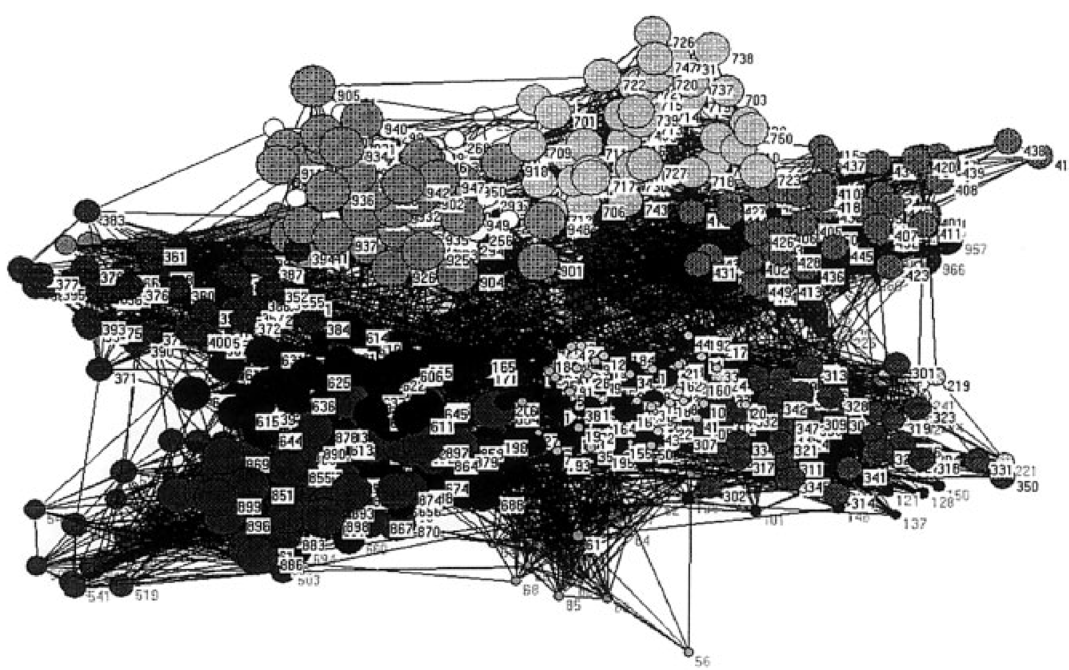
Our next concern is the statistical analysis of the networks that emerge in these three cases. The three networks are graphs, that is, the connections between nodes are undirected. The simplest Markov erg model for such entities is called ρ σ τ model by Frank and Strauss [35], who interpret these parameters as expressing three structural effects: density (ρ), clustering (σ), and transitivity (τ). The density effect expresses how much more or less likely than 50/50 are the odds that a tie is present vs. absent. A positive clustering effect (net of density) means that graphs with many "two-stars," that is, configurations in which one node is connected to two others, have higher probability than graphs with few two-stars. Holding constant density, a graph will have more two-stars if degree of the nodes, the number of contacts each has, has greater variance. The transitivity effect expresses the impact of triangles in which all three nodes in a triad are connected by a tie. A positive effect here means that net of density and variance in degree, a graph with more triangles is more likely than one with fewer

FIGURE 1



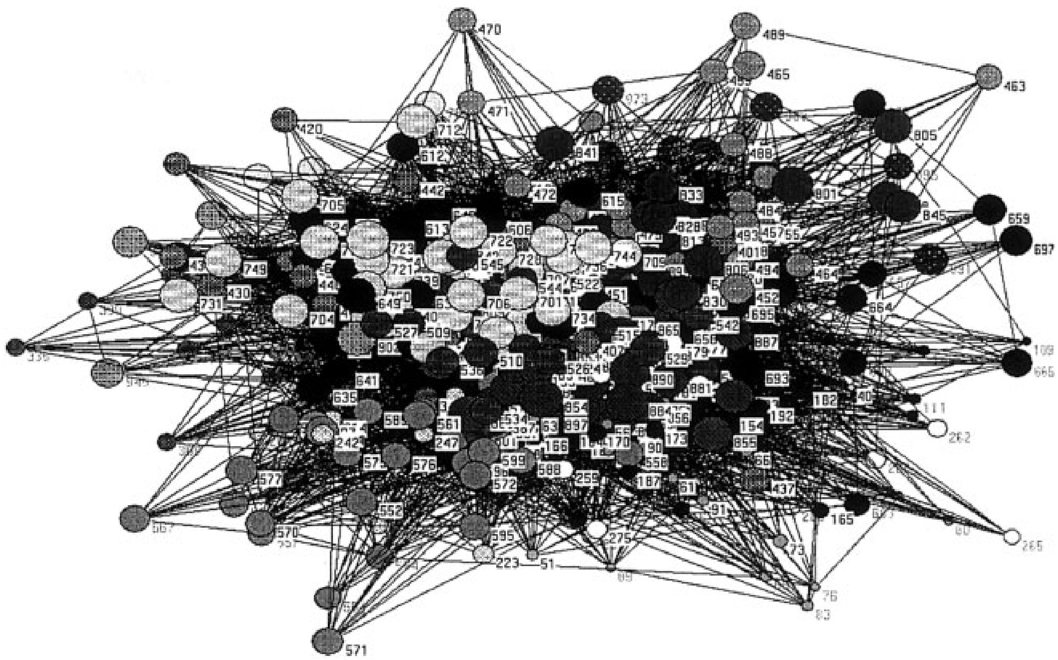
Neighborhood = 10, embeddedness = 0.90.

FIGURE 2



Neighborhood size = 50, embeddedness = 0.90.

FIGURE 3



Neighborhood size = 50, embeddedness = 0.50.

triangles, that is, the ties tend to arrange themselves in areas of locally high density.

Pseudo-likelihood estimation of the logit equation for this model provides the results found in Table 4. It must be cautioned that there is still much to learn about the robustness of the estimates. Standard errors and measures of fit must be regarded as nominal and indicative of likely order of magnitude and should not be used for inferential purposes. Also found in Table 4 are the results of estimating the same model on a real data set consisting of the reciprocated ties of political support in the US Senate during the 101st Congress, as revealed by two senators cosponsoring one another's legislation [36,37].

As Table 4 shows in all four cases, the density effect is negative, meaning that a tie is more likely to be absent than present. In all four networks, the clustering effect is negative, meaning that net of density, the degree distribution tends to be more homogeneous or, in other words, there is a tendency for ties to be more evenly distributed over the nodes rather than clustered on just a few of them. Also in all four cases, the transitivity effect is positive. This means that there would tend to be areas of high local density thus creating, net of overall density and degree dispersion, completely connected triples of

TABLE 2

Encounters per Tie

	10/0.90		50/0.90		50/0.50	
Encounters	Freq	%	Freq	%	Freq	%
1	2174	44.7	7298	85.2	9092	95.4
2	1223	25.2	1116	13.0	416	4.4
3	820	16.8	133	1.6	24	0.3
4	387	8.0	14	0.2	1	0.0
5	178	3.7	3	0.0	0	0.0
6	59	1.2	0	0.0	0	0.0
7	16	0.3	0	0.0	0	0.0
8	2	0.0	0	0.0	0	0.0

TABLE 3

Densities between Neighborhoods for Various Cut-off Values

Context	Density
10/0.90, Min = 1	0.181
50/0.90, Min = 5	0.589
50/0.50, Min = 5	1.000
10/0.90, Min = 2	0.020
50/0.90, Min = 10	0.047
50/0.50, Min = 10	1.000

TABLE 4

Estimates for the Basic Markov Graph Model

	Coefficient (SE)	Std Coefficient
10/0.90		
Density (ρ)	−2.859 (.183)	0.000
Clustering (σ)	−0.173 (0.10)	−0.225
Transitivity (τ)	1.356 (0.11)	+0.447
−2 Log L	% Improvement = 65%	
50/0.90		
Density (ρ)	−1.166 (.085)	0.000
Clustering (σ)	−0.121 (.003)	−0.343
Transitivity (τ)	0.794 (.005)	+0.485
−2 Log L	% Improvement = 36%	
50/0.50		
Density (ρ)	−3.155 (.070)	0.000
Clustering (σ)	−0.033 (.002)	−0.105
Transitivity (τ)	0.777 (.009)	+0.291
−2 Log L	% Improvement = 6%	
101st Senate		
Density (ρ)	−1.423 (.183)	0.000
Clustering (σ)	−0.030 (.004)	−0.389
Transitivity (τ)	0.182 (.008)	+1.148
−2 Log L	% Improvement = 26%	

actors. In all cases the magnitudes of the parameters, relative to their approximate standard errors, are large.

Two things appear to distinguish the cases in which cooperation emerges among strangers from the case in which it does not. First, the model fits better the first two cases. In the case of large neighborhoods and modest embeddedness, the improvement in fit (as measured by −2 Log of the Pseudolikelihood) is only 6%. Second, the standardized effects of both clustering and transitivity are lower in the third case compared to the first two. The tentative conclusion, therefore, is that network topologies in which degree dispersion is low, but there are areas of locally high density, are networks conducive to the evolution and maintenance of cooperation between strangers. The pattern in the US Senate data is more similar to the pattern in the first two network topologies than in the third. Therefore, we may also tentatively conclude that cooperation between strangers was viable in this particular milieu.

CONCLUSION

My aim in this article was to illustrate how new statistical models for social networks might be used to advance the agent-based modeling strategy in the study of complexity. Whether the conclusions about the statistical characterization of network topologies conducive to cooperation between strangers are correct or plausible or whether the coordination of this conclusion with actual data on political support has merit are less important issues than how one might coordinate the newly developed social network models with problems in complexity analysis.

Clearly much more work needs to be done. The formal bases of the pseudo-likelihood estimation procedure need to be researched. Alternative estimation techniques, such as Monte-Carlo Markov Chain estimation discussed by Snijders [38], need to be investigated. Simulation algorithms for exponential random graph models should be developed to explore more systematically the typical properties of the network topologies they create. Further coordination with other agent-based models and the network topologies they use should be encouraged.

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