NODE CONNECTIVITY AND CONDITIONAL DENSITY– D. White and F. Harary

Definitions of Cohesive Sets in Social Networks

The connectivity *K* (G) – kappa – of a connected graph G is the minimum number of nodes needed to disconnect G. Any graph with *K* or higher connectivity is said to be k-connected, where k= *K*. A k-component S of G is a maximal subgraph with *K* (S)=k. Sociologically, k-components are **cohesive** sets.

The edge-connectivity *K'* (G) – kappa prime – of a connected graph G is the minimum number of edges needed to disconnect G. Any graph with *K'* or higher connectivity is said to be k-edge-connected, where k= *K'*. A k-edge-component S of G is a maximal subgraph with *K'* (S)=k. Sociologically, k-components are **edge-cohesive** or **adhesive** sets.

By Menger's Theorem, a k-component of a graph G is a maximal set of nodes in which every pair is connected by at least k node-independent paths, and a k-edge-component of a graph G is a maximal set of nodes in which every pair is connected by at least k edge-independent paths.

The use of these definitions, pertaining to social cohesion in networks of interpersonal friendships, are exemplified in the following analysis of conflict and solidarity in a karate club, an excerpt of part IV of Douglas R. White and Frank Harary's article, "The Cohesiveness of Blocks in Social Networks: Node Connectivity and Conditional Density," *Sociological Methodology 2001*.

An Empirical Example: Zachary's Karate Club

Zachary's (1975, 1977) two-year ethnographic network study of 34 members of a karate club is a good proving ground to examine concepts, measures and hypotheses involving relational aspects of social solidarity. This section provides an illustration of how measures of cohesion are used to predict the outcome variable of sides taken in a factional dispute from the boundaries of nested cohesive sets. The disputants were the karate teacher (T, #1, Mr. Hi) and the club administrator (A, #34, John) and the dispute was about whether to improve the solvency of the club by raising fees (teacher) or by holding costs down (as A insisted). This resulted in each calling meetings at which they hoped to pass self-serving resolutions by encouraging attendance of their own supporters. The formation of factions was visible to the ethnographer and evident in meeting attendance, which varied in factional proportions according to the convener. Ultimately Mr. Hi (T) was fired, set up a separate club, and the factional split became the basis for each student's choice of which of the new clubs they would join. The prediction tested here is that when two "sociometric centers" of a group force a division into two, the cohesion measure will predict how members of the old group will distribute among the new ones.

A. Global View of the Karate Network

Figure 11 shows the network of friendships among the 34 members. Zachary weighted the strength of each friendship by the number of contexts (karate and other classes, hangouts, tournaments, and bars) in which the pair met, but the weights are not shown in the figure. Instead, concentric rings of adhesive subsets are circled in Figure 11 according to their 1-, 2-, 3-, and 4-edge connectivity. Table 3 lists the members and gives the number (n) in each set, the concentric nesting of the sets, and the edge connectivity k' of each. The set with highest adhesion, which consists of 10 members and includes A and T, is separable by four edges ($k' = 4$) but A and T are separated by a minimum of 10. There are, however, many different edge-cuts of size 10 that separate A and T. Hence unweighted edge-cuts, as well as adhesive sets, fail to predict faction membership. Each of the nested 1-, 2-, 3-, and 4-edge-connected subsets contains a cutnode (T) and lacks cohesivity since $k < 2$.

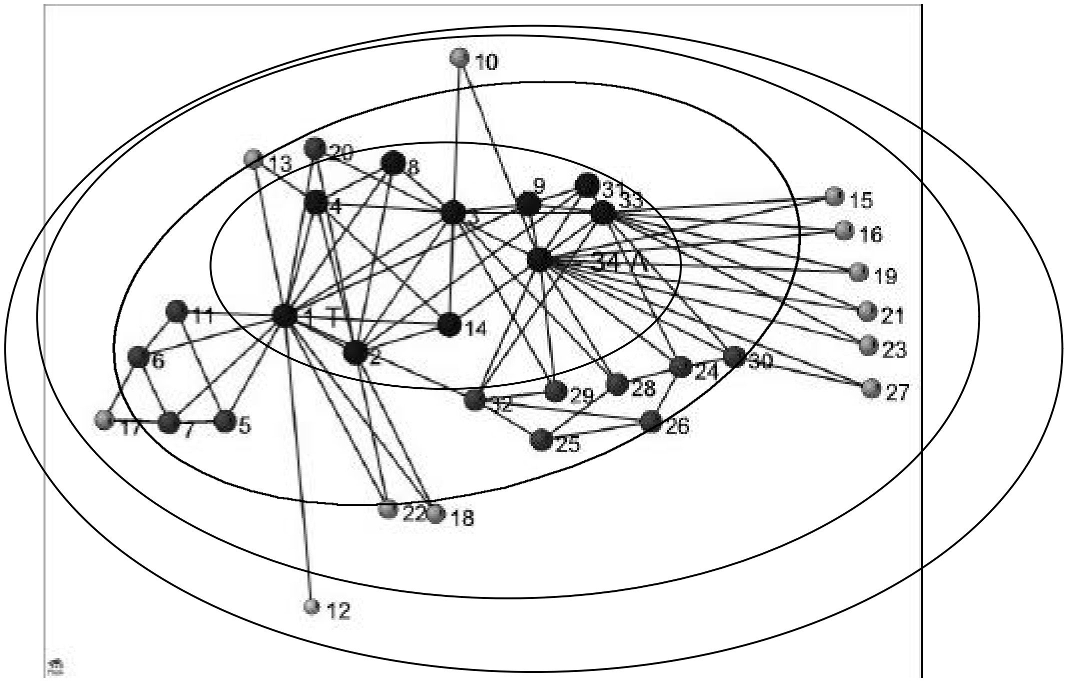

FIGURE 11. Nested adhesive sets for edge connectivities of 1, 2, 3, and 4. This and the following figures are drawn with Batagelj and Mrvar's (1997, 1998) Pajek software.

Sets	Members [1,34 leaders]	Nested in Set	*n*	*k*¢
1	1-34		34	1
2	1-11,13-34	1	33	2
3	1-9,11,14,20,24-26,28-34	2	22	3
4	1-4,8-9,14,31,33-34	3	10	4

TABLE 3
Edge Connectivity Sets for the Karate Club

Zachary used weighted minimum edge-cuts between A and T (the Local Ford-Fulkerson max flow-min cut theorem) to predict the separation of the two factions. Except for three persons who did not take sides, this gave a near-perfect prediction of the split. The particular distribution of weights on the edges, however, contributed to a unique-cut solution, pushed somewhat away from T since weights were highest for those close to him. Zachary did not utilize criteria for subgroup cohesion, but the dynamics of the dispute gives us the opportunity to examine cohesive blocks before the split and the role they played in mobilizing the taking of sides.

Looked at in terms of cohesion (Figure 12) the network has five cohesive blocks of connectivity 2 or greater, each enclosed in Figure 12 by one of the concentric circles. Only node T is common to them all. Two exclude node A (a 3-component within a 2-component) and three (a 4-component within a 3- within a 2-component) include node A.

Table 4 shows the cohesive blocks 1-5 circled in Figure 12, their members, number of nodes, the hierarchical nesting of each block, its connectivity, conditional density, and aggregate cohesion. An additional subset with the highest aggregate cohesion of 4.75 is also shown – the six people within the dotted circle in Figure 12 – which is not a maximal cohesive block but part of block 5 (with cohesion 4.24). This subset forms the core of support for Mr. Hi, while the remnants of block 5 after removing this subset, shown as set 7 in Table 4, are supporters of A.

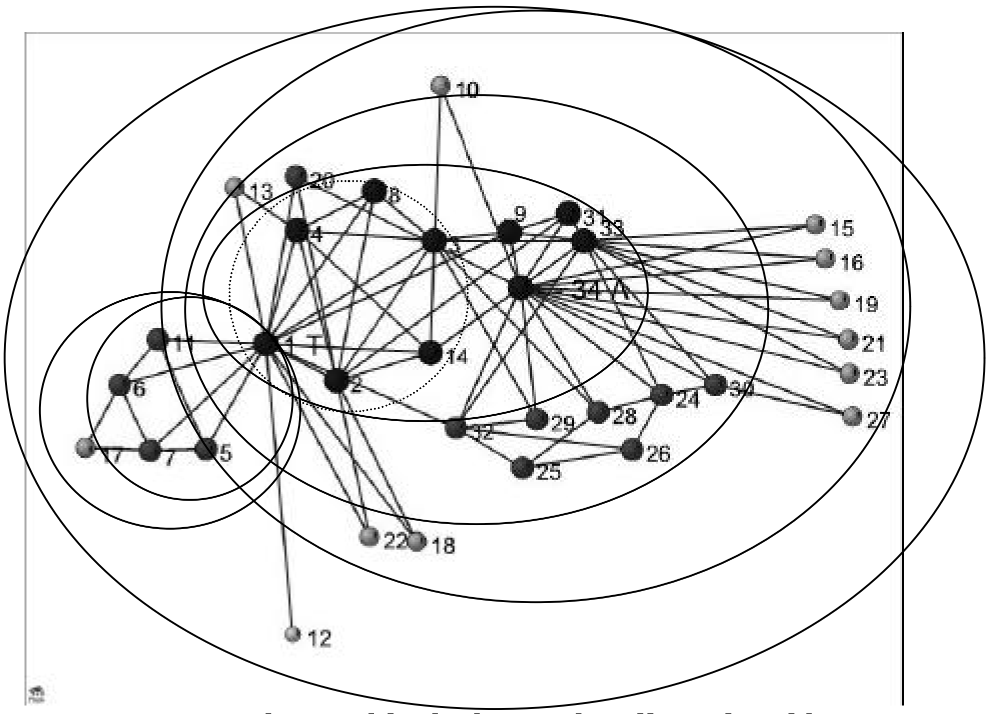

FIGURE 12. Cohesive blocks hierarchically ordered by connectivity into two nests (the outer dotted circle nests them all in a connected graph with connectivity 1).

Blocks and Sets	Members [1,34 leaders]	Nested in Set	*n*	*k*	r_2	Aggregate Cohesion
1	1,5-7,11,17		6	2	.20	2.20
2	1,5-7,11	2	6	3	.54	3.54
3	1-4,8-10,13-16,18-34		28	2	.12	2.12
4	1-4,8-9,14,20,24-26,28-34	3	18	3	.12	3.12
5	1-4,8-9,14,31,33,34	4	10	4	.24	4.24
6*	1-4,8,14	5	6	4	.75	4.75
7*	9,31,33,34	5	4	3	.00	3.00

* Sets 1-5 are cohesive blocks; set 6 is the densest cohesive subblock within 5 and set 7 is the residual within 5 after taking out set 6.

TABLE 4

Cohesive Block and Subset Characteristics for Karate Club

A first and approximate prediction of factions uses the number of node-independent paths (node-flow) joining pairs of nodes, and then takes the maximum spanning tree of the edges in the original network selected in order of largest number of node-independent paths (White and Newman 2001: the spanning tree portion of the algorithm, in general, breaks ties in favor of pairs of nodes separated by least distance). The result is depicted in Figure 13, in which the vertical line is a good predictor of the initial factional alignment, with followers of T to the left and those of A to the right. Person 9, on A's side of the prediction line, initially aligns with A but later switches to T. After removal of the cutnode between T and A, which is also person 9 (which also removes the two dotted lines in the figure), two trees remain with T and A at their respective centers. Except for those who do not take sides (three nodes labeled ?), and two others, 28 and 29, the trees predict the factional alignments. The spanning tree algorithm, however, introduces some noise to Figure 13 as a predictor variable because choice is arbitrary among edges that are equally well qualified for the spanning tree. Persons 28 and 29 are an example, and could equally well be linked by the algorithm to A, thereby improving the prediction.

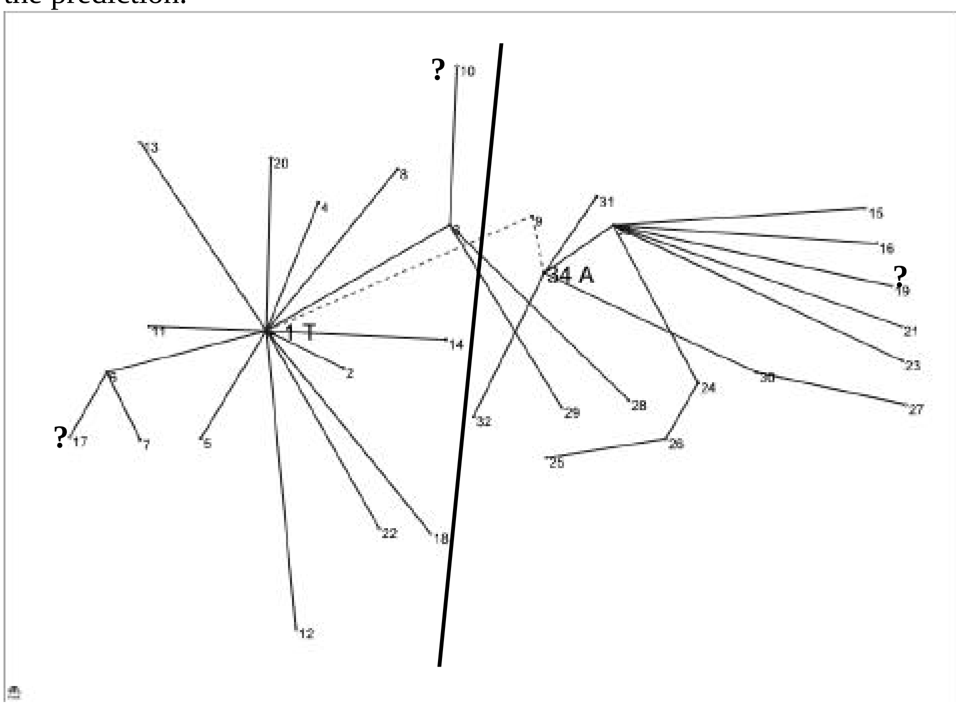

FIGURE 13. Maximum spanning tree of the numbers of node-independent paths between pairs of nodes, where solid lines predict faction members except for 28, 29 and nodes marked (?).

B. Closeup of Cohesion in the Karate Network

Because cohesive blocking is a deterministic procedure, it makes more precise predictions than White and Newman's approximation algorithm. If we situate the problem of determining factional divisions in the context of the opposition between leaders, as Zachary did, there are four persons – 9, 14, 20, and 32 – who had friendships with both leaders and thus had to make up their minds which leader to follow as the club split. Their membership in cohesive blocks and subsets provides a determinate prediction as to their decisions about which leader to follow. The choices they made corresponded not to the number of contexts in which they had friendships with T (Mr. Hi) or A (John), as Zachary would suggest, but to the pull of cohesive ties with others in core group of T (set 6 in Table 4) versus A (set 7). For each of these four people, who must decide between T and A, Table 5 contains four labeled rows: In the three columns under Mr. Hi's faction are the subset size (*n*), number of edges (*e*), and aggregate cohesion ($k + r_2$) within Mr. Hi’s faction (set 6); and similarly for A's faction, set 7. In the center of the table is a column that shows whether cohesion is greater with T (>) or A (<). In the rightmost columns are each person's predicted and actual choice of faction, showing that each of these people chose to align with the faction in which they have highest cohesion.

	Mr. Hi's Faction				A's Faction				
	n	*e*	**AC_1** $k + r_2$	**AC_1 AC_2**	*n*	*e*	**AC_2** $k + r_2$	**Predicted Choice of Faction**	**Actual Choice of Faction**
14	6	14	4.75	>	5	7	1.75	Mr. Hi	Mr. Hi
20	4	5	2.5	>	5	7	1.75	Mr. Hi	Mr. Hi
9	2	1	2.0	<	4	6	4.0	A	A
32	2	1	2.0	<	5	8	2.83	A	A

TABLE 5
Aggregate Cohesion (AC) with Leadership Factions for Persons Tied to Both Leaders and Obliged to Choose Between Them

Students 14 and 20, for example, had more cohesion with Mr. Hi's group than with A's, and they aligned with Mr. Hi’s faction in attendance at meetings. Students 9 and 32, on the other hand, had more cohesion with A's group and aligned with his faction. Each of these four people had to make a choice to drop a tie with the leader whose faction they rejected. If we remove the line connecting 14 to 34 (A) because 14 chose to join T's faction, for example, we observe in Figure 14 that even this one edge-removal results in a smaller 4-connected cohesive block of six persons {1, 2, 3, 4, 8, 14}, out of the original 10 in block 5, nested within a larger 3-cohesive block. All six persons in this 4-block align with T, as predicted from cohesion. If we allocate the remaining nodes in the 3-block according to their cohesion with T versus A, person 20 is predicted to go with T and the remainder with A. Allocating those in the 2- and 1-components by the same procedure, only person 10's alignment is indeterminate, and 10 was one of the three not factionally aligned.

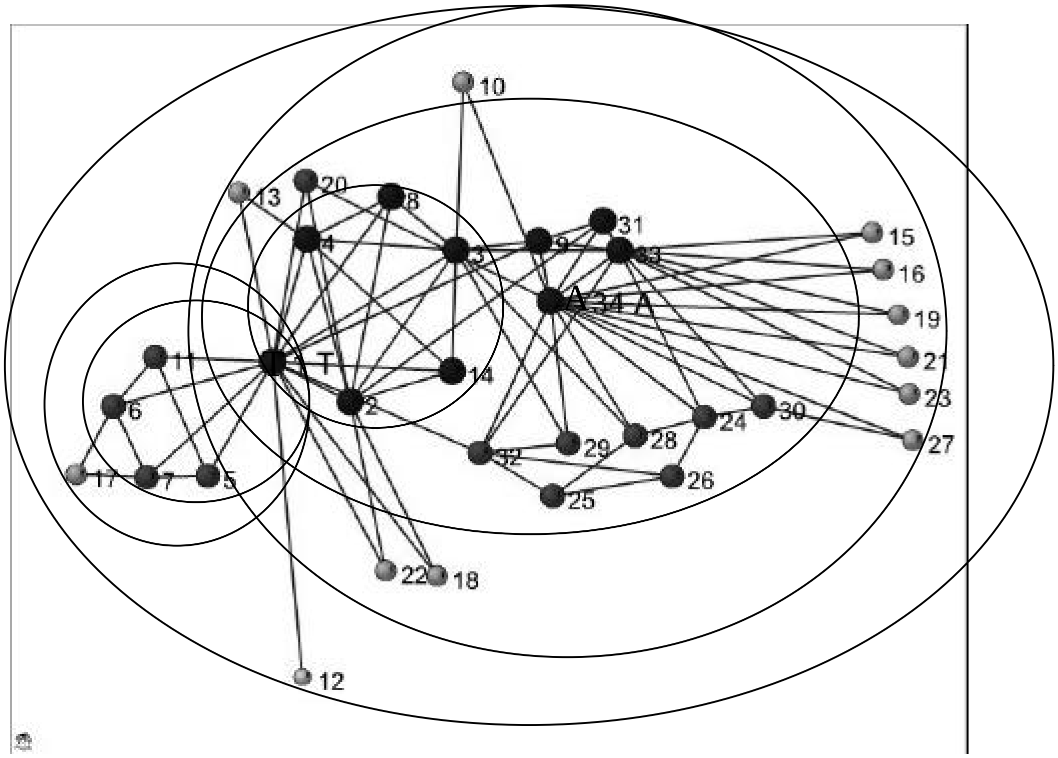

FIGURE 14. Nested cohesive sets by *k*-connectivity after person 14 affiliates with T.

The results of this test of the global predictions of faction membership from our connectivity measures are summarized in Table 6 (r = .969, p < .000001). The columns indicate whether cohesion is greater, equal, or less with Mr. Hi (T) than with John (A). This correlation is nearly identical to Zachary's prediction using the Ford-Fulkerson maximum flow-minimum cut algorithm on weighted edges, as shown in Table 7 (r = .955). Zachary's prediction, however, contains an unwarranted postulate of some kind of network "agency" that "finds" an optimal edge-cut without any explanation as to mechanism.

Cohesion Faction	Mr. Hi (T)	Equal for T and A	John (A)	Members by ID number
Mr. Hi's (T)	15			1-8,11-14,18,20,22
None	1	1	1	17, 10, 19
John's (A)			16	9,15,16,21,23-34

TABLE 6
Predictions of Faction Membership from Structural Cohesion (r = .969)

Edge-Cut Faction	Mr. Hi (T)	John (A)	Members by ID number
Mr. Hi's (T)	15		1-8,11-14,18,20,22
None	1	2	17, 10, 19
John's (A)		16	9,15,16,21,23-34

TABLE 7
Predictions of Faction Membership from Zachary's Minimum Weighted Edge-Cut (*r* = .955)

C. Evaluation of Results

Alignment of factional in the karate club is predicted by structural and path cohesion as well as by Zachary's adhesive weighted edge-cuts. The results shows the contribution of conditional density, in addition to cohesion measured by connectivity, to identifying localized high-density subgroups within cohesive blocks. The high-density subgroup (set 6) that was the core of T's support group was a very compact group with minimum distances among members, which may have contributed to leader T's retention of so many followers in the breakup of the club. Zachary had the right result as well, but possibly for the wrong reasons. Although both models make near-perfect predictions, the cohesion argument has advantages over the capacitated flow and possibly other arguments in the karate study on the grounds of parsimony (use of unweighted over weighted edges) and a process model of agency as the mechanism involved in segmentation.[1]

[1]The fact that Mr. Hi is the cutnode in a bifurcated network might help to explain – in sociological terms – why he is the instigator of the dispute in the first place: He has a set of at least five potential students who were never integrated into the larger cohesive block containing the administrator (#34), and for Mr. Hi it was clear from the beginning that they would follow his leadership. He was also a strong figure for many of his other adherents.